Stochastic differential equations with coefficients in Sobolev spaces

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Abstract

We consider Itô SDE $\mathrm{d}X_t = \sum_{j=1}^m A_j(X_t)\,\mathrm{d}w_t^j + A_0(X_t)\,\mathrm{d}t$ on $\mathbb{R}^d$. The diffusion coefficients $A_1, \cdots, A_m$ are supposed to be in the Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ with $p > d$, and to have linear growth; for the drift coefficient A_0, we consider two cases: (i) A_0 is continuous whose distributional divergence $\delta(A_0)$ w.r.t. the Gaussian measure γ_d exists, (ii) A_0 has the Sobolev regularity $W^{1,p'}_{\mathrm{loc}}$ for some $p' > 1$. Assume $\int_{\mathbb{R}^d} \exp\big[\lambda_0\big(|\delta(A_0)| + \sum_{j=1}^m(|\delta(A_j)|^2 + |\nabla A_j|^2)\big)\big]\mathrm{d}\gamma_d < +\infty$ for some $\lambda_0 > 0$, in the case (i), if the pathwise uniqueness of solutions holds, then the push-forward $(X_t)_\#\gamma_d$ admits a density with respect to γ_d. In particular, if the coefficients are bounded Lipschitz continuous, then X_t leaves the Lebesgue measure Leb_d quasi-invariant. In the case (ii), we develop a method used by G. Crippa and C. De Lellis for ODE and implemented by X. Zhang for SDE, to establish the existence and uniqueness of stochastic flow of maps.

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1 Introduction

Let $A_0, A_1, \ldots, A_m \colon \mathbb{R}^d \to \mathbb{R}^d$ be continuous vector fields on $\mathbb{R}^d$. We consider the following Itô stochastic differential equation on $\mathbb{R}^d$ (abbreviated as SDE)

$$\mathrm{d}X_t = \sum_{j=1}^m A_j(X_t)\,\mathrm{d}w_t^j + A_0(X_t)\,\mathrm{d}t, \quad X_0 = x, \tag{1.1}$$

where $w_t = (w_t^1, \ldots, w_t^m)$ is the standard Brownian motion on $\mathbb{R}^m$. It is a classical fact in the theory of SDE (see [16, 17, 21, 30]) that, if the coefficients A_j are globally Lipschitz continuous, then SDE (1.1) has a unique strong solution which defines a stochastic flow of homeomorphisms on $\mathbb{R}^d$; however contrary to ordinary differential equations (abbreviated as ODE), the regularity of the homeomorphisms is only Hölder continuity of order $0 < \alpha < 1$. Thus it is not clear whether the Lebesgue measure Leb_d on $\mathbb{R}^d$ admits a density under the flow X_t. In the case where the vector fields $A_j, j = 0, 1, \ldots, m$, are in $C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$, the SDE (1.1) defines a flow of diffeomorphisms, and Kunita [21] showed that the measures on $\mathbb{R}^d$ which have a strictly positive

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smooth density with respect to Leb_d are quasi-invariant under the flow. This result was recently generalized in [27] to the case where the drift A_0 is allowed to be only log-Lipschitz continuous. Studies on SDE beyond the Lipschitz setting attracted great interest during the last years, see for instance [10, 11, 13, 19, 20, 23, 24, 29, 34, 35].

In the context of ODE, existence of a flow of quasi-invariant measurable maps associated to a vector field A_0 belonging to Sobolev spaces appeared first in [6]. In the seminar paper [7], Di Perna and Lions developed transport equations to solve ODE without involving exponential integrability of $|\nabla A_0|$. On the other hand, L. Ambrosio [1] took advantage of using continuity equations which allowed him to construct quasi-invariant flows associated to vector fields A_0 with only BV regularity. In the framework for Gaussian measures, the Di Perna-Lions method was developed in [4], also in [2, 12] on the Wiener space.

The situation for SDE is quite different: even for the vector fields $A_0, A_1, \ldots, A_m$ in C^∞ with linear growth, if no conditions were imposed on the growth of the derivatives, the SDE (1.1) could not define a flow of diffeomorphisms (see [25, 26]). More precisely, let τ_x be the life time of the solution to (1.1) starting from x. The SDE (1.1) is said to be *complete* if for each $x \in \mathbb{R}^d$, $\mathbb{P}(\tau_x = +\infty) = 1$; it is said to be *strongly complete* if $\mathbb{P}(\tau_x = +\infty,\, x \in \mathbb{R}^d) = 1$. The goal in [26] is to construct examples for which the coefficients are smooth, but the SDE (1.1) is not strongly complete (see [11, 25] for positive examples). Now consider

$$\Sigma = \{(w, x) \in \Omega \times \mathbb{R}^d;\ \tau_x(w) = +\infty\}.$$

Suppose that the SDE (1.1) is complete, then for any probability measure μ on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \left(\int_\Omega \mathbf{1}_\Sigma(w, x)\, \mathrm{d}\mathbb{P}(w) \right) \mathrm{d}\mu(x) = 1.$$

By Fubini's theorem, $\int_\Omega \left(\int_{\mathbb{R}^d} \mathbf{1}_\Sigma(w, x)\, \mathrm{d}\mu(x) \right) \mathrm{d}\mathbb{P}(w) = 1$. It follows that there exists a full measure subset $\Omega_0 \subset \Omega$ such that for all $w \in \Omega_0$, $\tau_x(w) = +\infty$ holds for μ-almost every $x \in \mathbb{R}^d$. Now under the existence of a complete unique strong solution to SDE (1.1), we have a flow of measurable maps $x \to X_t(w, x)$.

Recently, inspired by a previous work due to Ambrosio, Lecumberry and Maniglia [3], Crippa and De Lellis [5] obtained some new type of estimates of perturbation for ODE whose coefficients have Sobolev regularity. More precisely, the absence of Lipschitz condition was filled by the following inequality: for $f \in W^{1,1}_{loc}(\mathbb{R}^d)$,

$$|f(x) - f(y)| \le C_d\, |x - y| \left(M_R|\nabla f|(x) + M_R|\nabla f|(y) \right)$$

holds for $x, y \in N^c$ and $|x - y| \le R$, where N is a negligible set of $\mathbb{R}^d$ and $M_R g$ is the maximal function defined by

$$M_R g(x) = \sup_{0<r\le R} \frac{1}{\mathrm{Leb}_d(B(x,r))} \int_{B(x,r)} |g(y)|\, \mathrm{d}y,$$

here $B(x, r) = \{y \in \mathbb{R}^d; |y - x| \le r\}$; the classical moment estimate was replaced by estimating the quantity

$$\int_{B(0,r)} \log \left(\frac{|X_t(x) - \tilde{X}_t(x)|}{\sigma} + 1 \right) \mathrm{d}x,$$

where $\sigma > 0$ is a small parameter. This method has recently been successfully implemented to SDE by X. Zhang in [36].

The aim in this paper is two-fold: first we shall study absolute continuity of the push-forward measure $(X_t)_\# \mathrm{Leb}_d$ with respect to Leb_d, once the SDE (1.1) has a unique strong solution;

secondly we shall construct strong solutions (for almost all initial values) using the approach mentioned above for SDE with coefficients in Sobolev space. The key point is to obtain *a priori* L^p estimate for the density. To this end, we shall work with the standard Gaussian measure γ_d; this will be done in Section 2. The main result in Section 3 is the following

Theorem 1.1. *Let $A_0, A_1, \ldots, A_m$ be continuous vector fields on $\mathbb{R}^d$ of linear growth. Assume that the diffusion coefficients $A_1, \ldots, A_m$ are in the Sobolev space $\cap_{q>1}\mathbb{D}_1^q(\gamma_d)$ and that $\delta(A_0)$ exists; furthermore there exists a constant $\lambda_0 > 0$ such that*

$$\int_{\mathbb{R}^d} \exp\left[\lambda_0\Big(|\delta(A_0)| + \sum_{j=1}^m \big(|\delta(A_j)|^2 + |\nabla A_j|^2\big)\Big)\right] \mathrm{d}\gamma_d < +\infty. \tag{1.2}$$

Suppose that pathwise uniqueness holds for SDE (1.1)*. Then $(X_t)_\# \gamma_d$ is absolutely continuous with respect to γ_d and the density is in the space $L^1 \log L^1$.*

A consequence of this theorem concerns the following classical situation.

Theorem 1.2. *Let $A_0, A_1, \ldots, A_m$ be globally Lipschitz continuous. Suppose that there exists a constant $C > 0$ such that*

$$\sum_{j=1}^m \langle x, A_j(x)\rangle^2 \le C\,(1 + |x|^2) \quad \textit{for all } x \in \mathbb{R}^d. \tag{1.3}$$

Then the stochastic flow of homeomorphisms X_t generated by SDE (1.1) *leaves the Lebesgue measure* Leb_d *quasi-invariant.*

Remark that the condition (1.3) not only includes the case of bounded Lipschitz diffusion coefficients, but also, maybe more significant, indicates the role of dispersion: the vector fields $A_1, \cdots, A_m$ should not go radically into infinity. The purpose of Section 4 is to find conditions that guarantee strict positivity of the density, in the case where the existence of the inverse flow is not known, see Theorem 4.4.

The main result in Section 5 is

Theorem 1.3. *Assume that the diffusion coefficients $A_1, \cdots A_m$ belong to the Sobolev space $\cap_{q>1}\mathbb{D}_1^q(\gamma_d)$ and the drift $A_0 \in \mathbb{D}_1^q(\gamma_d)$ for some $q > 1$. Assume* (1.2) *and that the coefficients $A_0, A_1, \cdots, A_m$ are of linear growth, then there is a unique stochastic flow of measurable maps $X : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, which solves* (1.1) *for almost all initial $x \in \mathbb{R}^d$ and the push-forward $(X_t(w,\cdot))_\# \gamma_d$ admits a density with respect to γ_d, which is in $L^1 \log L^1$.*

When the diffusion coefficients satisfy the uniform ellipticity, a classical result due to Stroock and Varadhan [32] says that if the diffusion coefficients $A_1, \cdots, A_m$ are bounded continuous and the drift A_0 is bounded Borel measurable, then the weak uniqueness holds, that is the uniqueness in law of the diffusion. This result was strengthened by Veretennikov [33], saying that in fact the pathwise uniqueness holds. When A_0 is not bounded, some conditions on diffusion coefficients were needed. In the case where the diffusion matrix $a = (a_{ij})$ is the identity, the drift A_0 in (1.1) can be quite singular: $A_0 \in L^p_{loc}(\mathbb{R}^d)$ with $p > d+2$ implies that the SDE (1.1) has the pathwise uniqueness (see Krylov-Röckner [20] for a more complete study); if the diffusion coefficients $A_1, \cdots, A_m$ are bounded continuous, under a Sobolev condition, namely, $A_j \in W^{1,2(d+1)}_{loc}$ for $j = 1, \cdots, m$ and $A_0 \in L^{2(d+1)}_{loc}(\mathbb{R}^d)$, X. Zhang proved in [34] that the SDE (1.1) admits a unique strong solution. Note that even in this uniformly non-degenerated case, if the diffusion coefficients lose the continuity, there are counterexamples for which the weak uniqueness does not hold, see [19, 31].

Finally we would like to mention that under weaker Sobolev type conditions, the connection between weak solutions and Fokker-Planck equations was investigated in [14, 22], some notions of "generalized solutions", as well as the phenomena of coalescence and splitting, were investigated in [23, 24]. Stochastic transport equations were studied in [15, 36].

2 L^p estimate of the density

The purpose of this section is to derive *a priori* estimates for the density; we assume that the coefficients $A_0, A_1, \ldots, A_m$ of SDE (1.1) are *smooth with compact support* in $\mathbb{R}^d$. Then the solution X_t, i.e., $x \mapsto X_t(x)$, is a stochastic flow of diffeomorphisms on $\mathbb{R}^d$. Moreover SDE (1.1) is equivalent to the following Stratonovich SDE

$$\mathrm{d}X_t = \sum_{j=1}^m A_j(X_t) \circ \mathrm{d}w_t^j + \tilde{A}_0(X_t)\,\mathrm{d}t, \quad X_0 = x, \tag{2.1}$$

where $\tilde{A}_0 = A_0 - \frac{1}{2}\sum_{j=1}^m \mathcal{L}_{A_j} A_j$ and $\mathcal{L}_A$ denotes the Lie derivative with respect to A.

Let γ_d be the standard Gaussian measure on $\mathbb{R}^d$, and $\gamma_t = (X_t)_\# \gamma_d$, $\tilde{\gamma}_t = (X_t^{-1})_\# \gamma_d$ the push-forwards of γ_d respectively by the flow X_t and its inverse flow X_t^{-1}. To fix ideas, we denote by $(\Omega, \mathscr{F}, \mathbb{P})$ the probability space on which the Brownian motion w_t is defined. Let $K_t = \frac{\mathrm{d}\gamma_t}{\mathrm{d}\gamma_d}$ and $\tilde{K}_t = \frac{\mathrm{d}\tilde{\gamma}_t}{\mathrm{d}\gamma_d}$ be the densities with respect to γ_d. By Lemma 4.3.1 in [21], the Radon-Nikodym derivative $\tilde{K}_t$ has the following explicit expression

$$\tilde{K}_t(x) = \exp\bigg(-\sum_{j=1}^m \int_0^t \delta(A_j)(X_s(x)) \circ \mathrm{d}w_s^j - \int_0^t \delta(\tilde{A}_0)(X_s(x))\,\mathrm{d}s \bigg), \tag{2.2}$$

where $\delta(A_j)$ denotes the divergence of A_j with respect to the Gaussian measure γ_d:

$$\int_{\mathbb{R}^d} \langle \nabla\varphi, A_j\rangle\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d} \varphi\,\delta(A_j)\,\mathrm{d}\gamma_d, \quad \varphi \in C_c^1(\mathbb{R}^d).$$

It is easy to see that K_t and $\tilde{K}_t$ are related to each other by the equality below:

$$K_t(x) = \big[\tilde{K}_t\big(X_t^{-1}(x)\big)\big]^{-1}. \tag{2.3}$$

In fact, for any $\psi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \psi(x)\,\mathrm{d}\gamma_d(x) &= \int_{\mathbb{R}^d} \psi\big[X_t\big(X_t^{-1}(x)\big)\big]\,\mathrm{d}\gamma_d(x) \\ &= \int_{\mathbb{R}^d} \psi[X_t(y)]\tilde{K}_t(y)\,\mathrm{d}\gamma_d(y) = \int_{\mathbb{R}^d} \psi(x)\tilde{K}_t\big(X_t^{-1}(x)\big)K_t(x)\,\mathrm{d}\gamma_d(x),\end{aligned}$$

which leads to (2.3) due to the arbitrariness of $\psi \in C_c^\infty(\mathbb{R}^d)$. In the following we shall estimate the $L^p(\mathbb{P} \times \gamma_d)$ norm of K_t.

We rewrite the density (2.2) with the Itô integral:

$$\tilde{K}_t(x) = \exp\bigg(-\sum_{j=1}^m \int_0^t \delta(A_j)(X_s(x))\,\mathrm{d}w_s^j - \int_0^t \bigg[\frac{1}{2}\sum_{j=1}^m \mathcal{L}_{A_j}\delta(A_j) + \delta(\tilde{A}_0)\bigg](X_s(x))\,\mathrm{d}s \bigg). \tag{2.4}$$

Lemma 2.1. *We have*

$$\frac{1}{2}\sum_{j=1}^{m}\mathcal{L}_{A_j}\delta(A_j)+\delta(\tilde{A}_0)=\delta(A_0)+\frac{1}{2}\sum_{j=1}^{m}|A_j|^2+\frac{1}{2}\sum_{j=1}^{m}\langle\nabla A_j,(\nabla A_j)^*\rangle, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ *denotes the inner product of* $\mathbb{R}^d\otimes\mathbb{R}^d$ *and* $(\nabla A_j)^*$ *the transpose of* ∇A_j.

Proof. Let A be a C^2 vector field on $\mathbb{R}^d$. From the expression

$$\delta(A)=\sum_{k=1}^{d}\Big(x_kA^k-\frac{\partial A^k}{\partial x_k}\Big),$$

we get

$$\mathcal{L}_A\delta(A)=\sum_{\ell,k=1}^{d}\Big(A^\ell A^k\delta_{k\ell}+A^\ell x_k\frac{\partial A^k}{\partial x_\ell}-A^\ell\frac{\partial^2A^k}{\partial x_\ell\partial x_k}\Big). \tag{2.6}$$

Note that

$$\frac{\partial}{\partial x_k}\Big(A^\ell\frac{\partial A^k}{\partial x_\ell}\Big)=\frac{\partial A^k}{\partial x_\ell}\frac{\partial A^\ell}{\partial x_k}+A^\ell\frac{\partial^2A^k}{\partial x_k\partial x_\ell}.$$

Thus, by means of (2.6), we obtain

$$\mathcal{L}_A\delta(A)=|A|^2+\delta(\mathcal{L}_AA)+\langle\nabla A,(\nabla A)^*\rangle. \tag{2.7}$$

Recall that $\delta(\tilde{A}_0)=\delta(A_0)-\frac{1}{2}\sum_{j=1}^m\delta(\mathcal{L}_{A_j}A_j)$. Hence, replacing A by A_j in (2.7) and summing over j, gives formula (2.5). □

We can now prove the following key estimate.

Theorem 2.2. *For* $p>1$,

$$\|K_t\|_{L^p(\mathbb{P}\times\gamma_d)}\leq\Big[\int_{\mathbb{R}^d}\exp\Big(pt\Big[2|\delta(A_0)|+\sum_{j=1}^{m}\big(|A_j|^2+|\nabla A_j|^2+2(p-1)|\delta(A_j)|^2\big)\Big]\Big)\,\mathrm{d}\gamma_d\Big]^{\frac{p-1}{p(2p-1)}}. \tag{2.8}$$

Proof. Using relation (2.3), we have

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}[K_t^p(x)]\,\mathrm{d}\gamma_d(x)&=\mathbb{E}\int_{\mathbb{R}^d}\big[\tilde{K}_t\big(X_t^{-1}(x)\big)\big]^{-p}\,\mathrm{d}\gamma_d(x)\\&=\mathbb{E}\int_{\mathbb{R}^d}\big[\tilde{K}_t(y)\big]^{-p}\tilde{K}_t(y)\,\mathrm{d}\gamma_d(y)\\&=\int_{\mathbb{R}^d}\mathbb{E}\big[\big(\tilde{K}_t(x)\big)^{-p+1}\big]\,\mathrm{d}\gamma_d(x).\end{aligned} \tag{2.9}$$

To simplify the notation, denote the right hand side of (2.5) by Φ. Then $\tilde{K}_t(x)$ rewrites as

$$\tilde{K}_t(x)=\exp\Big(-\sum_{j=1}^{m}\int_0^t\delta(A_j)(X_s(x))\,\mathrm{d}w_s^j-\int_0^t\Phi(X_s(x))\,\mathrm{d}s\Big).$$

Fixing an arbitrary $r>0$, we get

$$
\begin{aligned}
\big(\tilde K_t(x)\big)^{-r} &= \exp\bigg(r\sum_{j=1}^m\int_0^t \delta(A_j)(X_s(x))\,\mathrm{d}w_s^j + r\int_0^t \Phi(X_s(x))\,\mathrm{d}s\bigg)\\
&= \exp\bigg(r\sum_{j=1}^m\int_0^t \delta(A_j)(X_s(x))\,\mathrm{d}w_s^j - r^2\sum_{j=1}^m\int_0^t \big|\delta(A_j)(X_s(x))\big|^2\,\mathrm{d}s\bigg)\\
&\quad\times \exp\bigg(\int_0^t \Big(r^2\sum_{j=1}^m|\delta(A_j)|^2 + r\Phi\Big)(X_s(x))\,\mathrm{d}s\bigg).
\end{aligned}
$$

By Cauchy-Schwarz's inequality,

$$
\begin{aligned}
\mathbb{E}\big[\big(\tilde K_t(x)\big)^{-r}\big] &\le \bigg[\mathbb{E}\exp\bigg(2r\sum_{j=1}^m\int_0^t \delta(A_j)(X_s(x))\,\mathrm{d}w_s^j - 2r^2\sum_{j=1}^m\int_0^t \big|\delta(A_j)(X_s(x))\big|^2\,\mathrm{d}s\bigg)\bigg]^{1/2}\\
&\quad\times\bigg[\mathbb{E}\exp\bigg(\int_0^t\Big(2r^2\sum_{j=1}^m|\delta(A_j)|^2+2r\Phi\Big)(X_s(x))\,\mathrm{d}s\bigg)\bigg]^{1/2}\\
&= \bigg[\mathbb{E}\exp\bigg(\int_0^t\Big(2r^2\sum_{j=1}^m|\delta(A_j)|^2+2r\Phi\Big)(X_s(x))\,\mathrm{d}s\bigg)\bigg]^{1/2}, \qquad (2.10)
\end{aligned}
$$

since the first term on the right hand side of the inequality in (2.10) is the expectation of a martingale. Let

$$
\tilde\Phi_r = 2r|\delta(A_0)| + r\sum_{j=1}^m\big(|A_j|^2+|\nabla A_j|^2+2r|\delta(A_j)|^2\big).
$$

Then by (2.10), along with the definition of Φ and Cauchy-Schwarz's inequality, we obtain

$$
\int_{\mathbb{R}^d}\mathbb{E}\big[\big(\tilde K_t(x)\big)^{-r}\big]d\gamma_d \le \bigg[\int_{\mathbb{R}^d}\mathbb{E}\exp\bigg(\int_0^t\tilde\Phi_r(X_s(x))\,\mathrm{d}s\bigg)\mathrm{d}\gamma_d\bigg]^{1/2}. \qquad (2.11)
$$

Following the idea of A.B. Cruzeiro ([6] Corollary 2.2, see also Theorem 7.3 in [8]) and by Jensen's inequality,

$$
\exp\bigg(\int_0^t\tilde\Phi_r(X_s(x))\,\mathrm{d}s\bigg) = \exp\bigg(\int_0^t t\,\tilde\Phi_r(X_s(x))\,\frac{\mathrm{d}s}{t}\bigg) \le \frac1t\int_0^t e^{t\,\tilde\Phi_r(X_s(x))}\,\mathrm{d}s.
$$

Define $I(t)=\sup_{0\le s\le t}\int_{\mathbb{R}^d}\mathbb{E}[K_t^p(x)]\,\mathrm{d}\gamma_d$. Integrating on both sides of the above inequality and by Hölder's inequality,

$$
\begin{aligned}
\int_{\mathbb{R}^d}\mathbb{E}\exp\bigg(\int_0^t\tilde\Phi_r(X_s(x))\,\mathrm{d}s\bigg)\mathrm{d}\gamma_d(x) &\le \frac1t\int_0^t\mathbb{E}\int_{\mathbb{R}^d}e^{t\,\tilde\Phi_r(X_s(x))}\,\mathrm{d}\gamma_d(x)\,\mathrm{d}s\\
&= \frac1t\int_0^t\mathbb{E}\int_{\mathbb{R}^d}e^{t\,\tilde\Phi_r(y)}K_s(y)\,\mathrm{d}\gamma_d(y)\,\mathrm{d}s\\
&\le \frac1t\int_0^t\big\|e^{t\,\tilde\Phi_r}\big\|_{L^q(\gamma_d)}\|K_s\|_{L^p(\mathbb{P}\times\gamma_d)}\,\mathrm{d}s\\
&\le \big\|e^{t\,\tilde\Phi_r}\big\|_{L^q(\gamma_d)}\,I(t)^{1/p},
\end{aligned}
$$

where q is the conjugate number of p. Thus it follows from (2.11) that

$$\int_{\mathbb{R}^d} \mathbb{E}\big[\big(\tilde K_t(x)\big)^{-r}\big]\,\mathrm{d}\gamma_d(x) \le \big\|e^{t\,\tilde\Phi_r}\big\|_{L^q(\gamma_d)}^{1/2}\, I(t)^{1/2p}. \tag{2.12}$$

Taking $r = p-1$ in the above estimate and by (2.9), we obtain

$$\int_{\mathbb{R}^d} \mathbb{E}[K_t^p(x)]\,\mathrm{d}\gamma_d(x) \le \big\|e^{t\,\tilde\Phi_{p-1}}\big\|_{L^q(\gamma_d)}^{1/2}\, I(t)^{1/2p}.$$

Thus we have $I(t) \le \big\|e^{t\,\tilde\Phi_{p-1}}\big\|_{L^q(\gamma_d)}^{1/2}\, I(t)^{1/2p}$. Solving this inequality for $I(t)$ gives

$$\int_{\mathbb{R}^d} \mathbb{E}[K_t^p(x)]\,\mathrm{d}\gamma_d(x) \le I(t) \le \bigg[\int_{\mathbb{R}^d} \exp\bigg(\frac{pt}{p-1}\tilde\Phi_{p-1}(x)\bigg)\,\mathrm{d}\gamma_d(x)\bigg]^{\frac{p-1}{2p-1}}.$$

Now the desired estimate follows from the definition of $\tilde\Phi_{p-1}$. □

Corollary 2.3. *For any $p > 1$,*

$$\|\tilde K_t\|_{L^p(\mathbb{P}\times\gamma_d)} \le \Bigg[\int_{\mathbb{R}^d} \exp\Bigg((p+1)t\bigg[2|\delta(A_0)| + \sum_{j=1}^m \big(|A_j|^2 + |\nabla A_j|^2 + 2p|\delta(A_j)|^2\big)\bigg]\Bigg)\,\mathrm{d}\gamma_d\Bigg]^{\frac{1}{2p+1}}. \tag{2.13}$$

Proof. Similar to (2.12), we have for $r > 0$,

$$\int_{\mathbb{R}^d} \mathbb{E}\big[\big(\tilde K_t(x)\big)^{r}\big]\,\mathrm{d}\gamma_d(x) \le \big\|e^{t\,\tilde\Phi_r}\big\|_{L^q(\gamma_d)}^{1/2}\, I(t)^{1/2p}, \tag{2.14}$$

where $\tilde\Phi_r$ and $I(t)$ are defined as above. Since $I(t) \le \big\|e^{t\,\tilde\Phi_{p-1}}\big\|_{L^q(\gamma_d)}^{p/(2p-1)}$, by taking $r = p-1$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}\big[\big(\tilde K_t(x)\big)^{p-1}\big]\,\mathrm{d}\gamma_d(x) &\le \big\|e^{t\,\tilde\Phi_{p-1}}\big\|_{L^q(\gamma_d)}^{p/(2p-1)}\\ &= \bigg[\int_{\mathbb{R}^d} \exp\bigg(pt\Big[2|\delta(A_0)| + \sum_{j=1}^m \big(|A_j|^2 + |\nabla A_j|^2 + 2(p-1)|\delta(A_j)|^2\big)\Big]\bigg)\,\mathrm{d}\gamma_d\bigg]^{\frac{p-1}{2p-1}}.\end{aligned}$$

Replacing p by $p+1$ in the last inequality gives the claimed estimate. □

3 Absolute continuity under flows generated by SDEs

Now assume that the coefficients A_j in SDE (1.1) are *continuous* and of linear growth. Then it is well known that SDE (1.1) has a weak solution of infinite life time. In order to apply the results of the preceding section, we shall regularize the vector fields using the Ornstein-Uhlenbeck semigroup $\{P_\varepsilon\}_{\varepsilon>0}$ on $\mathbb{R}^d$:

$$P_\varepsilon A(x) = \int_{\mathbb{R}^d} A\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y\big)\,\mathrm{d}\gamma_d(y).$$

We have the following simple properties.

Lemma 3.1. *Assume that A is continuous and $|A(x)| \le C\,(1+|x|^q)$ for some $q \ge 0$. Then*

(i) *there is $C_q > 0$ independent of ε, such that*

$$|P_\varepsilon A(x)| \le C_q\,(1+|x|^q), \quad \textit{for all } x \in \mathbb{R}^d;$$

(ii) *$P_\varepsilon A$ converges uniformly to A on any compact subset as $\varepsilon \to 0$.*

Proof. (i) Note that $\big|e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y\big| \le |x| + |y|$ and that there exists a constant $C > 0$ such that $(|x|+|y|)^q \le C\,(|x|^q + |y|^q)$. Using the growth condition on A, we have for some constant $C > 0$ (depending on q),

$$\begin{aligned}|P_\varepsilon A(x)| &\le \int_{\mathbb{R}^d} \big|A\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y\big)\big| \mathrm{d}\gamma_d(y) \\ &\le C \int_{\mathbb{R}^d} \big(1+|x|^q+|y|^q\big)\mathrm{d}\gamma_d(y) \le C\,\big(1+|x|^q+M_q\big)\end{aligned}$$

where $M_q = \int_{\mathbb{R}^d} |y|^q\,\mathrm{d}\gamma_d(y)$. Changing the constant yields (i).

(ii) Fix $R > 0$ and x in the closed ball $B(R)$ of radius R, centered at 0. Let $R_1 > R$ be arbitrary. We have

$$\begin{aligned}|P_\varepsilon A(x) - A(x)| &\le \int_{\mathbb{R}^d} \big|A\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y\big) - A(x)\big| \mathrm{d}\gamma_d(y) \\ &= \bigg(\int_{B(R_1)} + \int_{B(R_1)^c}\bigg)\big|A\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y\big) - A(x)\big|\,\mathrm{d}\gamma_d(y) \\ &=: I_1 + I_2. \end{aligned} \tag{3.1}$$

By the growth condition on A, for some constant $C_q > 0$, independent of ε, we have

$$\begin{aligned} I_2 &\le \int_{B(R_1)^c} \Big(\big|A\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y\big)\big| + |A(x)|\Big)\,\mathrm{d}\gamma_d(y) \\ &\le C_q \int_{B(R_1)^c} \big(1 + R^q + |y|^q\big)\,\mathrm{d}\gamma_d(y), \end{aligned}$$

where the last term tends to 0 as $R_1 \to +\infty$. For given $\eta > 0$, we may take R_1 large enough such that $I_2 < \eta$. Then there exists $\varepsilon_{R_1} > 0$ such that for $\varepsilon < \varepsilon_{R_1}$ and $|y| \le R_1$,

$$\big|e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y\big| \le e^{-\varepsilon}R + \sqrt{1-e^{-2\varepsilon}}\,R_1 \le R_1.$$

Note that

$$\big|e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y - x\big| \le \varepsilon R + \sqrt{2\varepsilon}R_1, \quad \text{for } |x| \le R,\ |y| \le R_1.$$

Since A is uniformly continuous on $B(R_1)$, there exits $\varepsilon_0 \le \varepsilon_{R_1}$ such that

$$\big|A(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\,y) - A(x)\big| \le \eta \quad \text{for all } y \in B(R_1),\ \varepsilon \le \varepsilon_0.$$

As a result, the term $I_1 \le \eta$. Therefore by (3.1), for any $\varepsilon \le \varepsilon_0$,

$$\sup_{|x|\le R} |P_\varepsilon A(x) - A(x)| \le 2\eta.$$

The result follows from the arbitrariness of $\eta > 0$. □

The vector field $P_\varepsilon A$ is smooth on $\mathbb{R}^d$ but does not have compact support. We introduce cut-off functions $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d, [0,1])$ satisfying

$$\varphi_\varepsilon(x) = 1 \text{ if } |x| \le \frac{1}{\varepsilon}, \quad \varphi_\varepsilon(x) = 0 \text{ if } |x| \ge \frac{1}{\varepsilon} + 2 \quad \text{and } \|\nabla\varphi_\varepsilon\|_\infty \le 1.$$

Set

$$A_j^\varepsilon = \varphi_\varepsilon P_\varepsilon A_j,\ j = 0, 1, \dots, m.$$

Now consider the Itô SDE (1.1) with A_j being replaced by A_j^ε ($j = 0, 1, \dots, m$), and denote the corresponding terms by adding the superscript ε, e.g. X_t^ε, K_t^ε, etc.

In the sequel, we shall give a uniform estimate to K_t^ε. To this end, we need some preparations in the spirit of Malliavin calculus [28]. For a vector field A on $\mathbb{R}^d$ and $p > 1$, we say that $A \in \mathbb{D}_1^p(\gamma_d)$ if $A \in L^p(\gamma_d)$ and if there exists $\nabla A\colon \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ in $L^p(\gamma_d)$ such that for any $v \in \mathbb{R}^d$,

$$\nabla A(x)(v) = \partial_v A := \lim_{\eta \to 0} \frac{A(x + \eta v) - A(x)}{\eta} \quad \text{holds in } L^{p'}(\gamma_d) \text{ for any } p' < p.$$

For such $A \in \mathbb{D}_1^p(\gamma_d)$, the divergence $\delta(A) \in L^p(\gamma_d)$ exists and the following relations hold:

$$\nabla P_\varepsilon A = e^{-\varepsilon} P_\varepsilon(\nabla A), \quad \delta(P_\varepsilon A) = e^\varepsilon P_\varepsilon(\delta(A)). \tag{3.2}$$

If $A \in L^p(\gamma_d)$, then $P_\varepsilon A \in \mathbb{D}_1^p(\gamma_d)$ and $\lim_{\varepsilon \to 0} \|P_\varepsilon A - A\|_{L^p} = 0$.

Lemma 3.2. *Assume the vector field $A \in \mathbb{D}_1^p(\gamma_d)$ with $p > 1$, and denote by $A^\varepsilon = \varphi_\varepsilon P_\varepsilon A$. Then for $\varepsilon \in]0, 1]$,*

$$\begin{aligned}
|\delta(A^\varepsilon)| &\le P_\varepsilon\big(|A| + e|\delta(A)|\big), \\
|A^\varepsilon|^2 &\le P_\varepsilon\big(|A|^2\big), \\
|\nabla A^\varepsilon|^2 &\le P_\varepsilon\big[2\big(|A|^2 + |\nabla A|^2\big)\big], \\
|\delta(A^\varepsilon)|^2 &\le P_\varepsilon\big[2\big(|A|^2 + e^2|\delta(A)|^2\big)\big].
\end{aligned}$$

Proof. Note that according to (3.2), $\delta(A^\varepsilon) = \delta(\varphi_\varepsilon P_\varepsilon A) = \varphi_\varepsilon e^\varepsilon P_\varepsilon \delta(A) - \langle \nabla \varphi_\varepsilon, P_\varepsilon A\rangle$, from where the first inequality follows. In the same way, the other results are obtained. □

Applying Theorem 2.2 to K_t^ε with $p = 2$, we have

$$\|K_t^\varepsilon\|_{L^2(\mathbb{P} \times \gamma_d)} \le \left[\int_{\mathbb{R}^d} \exp\left(2t\Big[2|\delta(A_0^\varepsilon)| + \sum_{j=1}^m \big(|A_j^\varepsilon|^2 + |\nabla A_j^\varepsilon|^2 + 2|\delta(A_j^\varepsilon)|^2\big)\Big]\right) \mathrm{d}\gamma_d\right]^{1/6}. \tag{3.3}$$

By Lemma 3.2,

$$\begin{aligned}
&2\,|\delta(A_0^\varepsilon)| + \sum_{j=1}^m \big(|A_j^\varepsilon|^2 + |\nabla A_j^\varepsilon|^2 + 2|\delta(A_j^\varepsilon)|^2\big) \\
&\qquad \le P_\varepsilon\left[2|A_0| + 2e|\delta(A_0)| + \sum_{j=1}^m \big(7|A_j|^2 + 2|\nabla A_j|^2 + 4e^2|\delta(A_j)|^2\big)\right].
\end{aligned}$$

We deduce from Jensen's inequality and the invariance of γ_d under the action of the semigroup P_ε that

$$\|K_t^\varepsilon\|_{L^2(\mathbb{P} \times \gamma_d)} \le \left[\int_{\mathbb{R}^d} \exp\left(4t\Big[|A_0| + e|\delta(A_0)| + \sum_{j=1}^m \big(4|A_j|^2 + |\nabla A_j|^2 + 2e^2|\delta(A_j)|^2\big)\Big]\right) \mathrm{d}\gamma_d\right]^{1/6} \tag{3.4}$$

for any $\varepsilon \le 1$. According to (3.4), we consider the following conditions.

Assumptions (H):

(A1) For $j = 1, \ldots, m$, $A_j \in \cap_{q \geq 1} \mathbb{D}_1^q(\gamma_d)$, A_0 is continuous and $\delta(A_0)$ exists.

(A2) The vector fields $A_0, A_1, \ldots, A_m$ have linear growth.

(A3) There exists $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^d} \exp\left[\lambda_0\left(|\delta(A_0)| + \sum_{j=1}^m |\delta(A_j)|^2\right)\right] \mathrm{d}\gamma_d < +\infty.$$

(A4) There exists $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^d} \exp\left(\lambda_0 \sum_{j=1}^m |\nabla A_j|^2\right) \mathrm{d}\gamma_d < +\infty.$$

Note that by Sobolev's embedding theorem, the diffusion coefficients $A_1, \ldots, A_m$ admit Hölder continuous versions. In what follows, we consider these continuous versions. It is clear that under the conditions (A2)–(A4), there exists $T_0 > 0$ small enough, such that

$$\Lambda_{T_0} := \left[\int_{\mathbb{R}^d} \exp\left(4T_0\Big[|A_0| + e|\delta(A_0)| + \sum_{j=1}^m \left(4|A_j|^2 + |\nabla A_j|^2 + 2e^2|\delta(A_j)|^2\right)\Big]\right) \mathrm{d}\gamma_d\right]^{1/6} < \infty. \quad (3.5)$$

In this case, for $t \in [0, T_0]$,

$$\sup_{0<\varepsilon\leq 1} \|K_t^\varepsilon\|_{L^2(\mathbb{P}\times\gamma_d)} \leq \Lambda_{T_0}. \quad (3.6)$$

Theorem 3.3. *Let $T > 0$ be given. Under* (A1)–(A4) *in Assumptions* **(H)**, *there are two positive constants C_1 and C_2, independent of ε, such that*

$$\sup_{0<\varepsilon\leq 1} \mathbb{E}\int_{\mathbb{R}^d} K_t^\varepsilon |\log K_t^\varepsilon| \, \mathrm{d}\gamma_d \leq 2\,(C_1 T)^{1/2} \Lambda_{T_0} + C_2 T \Lambda_{T_0}^2, \quad \textit{for all } t \in [0, T].$$

Proof. We follow the arguments of Proposition 4.4 in [12]. By (2.3) and (2.4), we have

$$K_t^\varepsilon(X_t^\varepsilon(x)) = \left[\tilde{K}_t^\varepsilon(x)\right]^{-1} = \exp\left(\sum_{j=1}^m \int_0^t \delta(A_j^\varepsilon)(X_s^\varepsilon(x)) \, \mathrm{d}w_s^j + \int_0^t \Phi_\varepsilon(X_s^\varepsilon(x)) \, \mathrm{d}s\right),$$

where

$$\Phi_\varepsilon = \delta(A_0^\varepsilon) + \frac{1}{2}\sum_{j=1}^m |A_j^\varepsilon|^2 + \frac{1}{2}\sum_{j=1}^m \langle \nabla A_j^\varepsilon, (\nabla A_j^\varepsilon)^* \rangle.$$

Thus

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d} K_t^\varepsilon |\log K_t^\varepsilon| \, \mathrm{d}\gamma_d = \mathbb{E}\int_{\mathbb{R}^d} \big|\log K_t^\varepsilon(X_t^\varepsilon(x))\big| \, \mathrm{d}\gamma_d(x) \\
&\quad \leq \mathbb{E}\int_{\mathbb{R}^d} \left|\sum_{j=1}^m \int_0^t \delta(A_j^\varepsilon)(X_s^\varepsilon(x)) \, \mathrm{d}w_s^j\right| \mathrm{d}\gamma_d(x) + \mathbb{E}\int_{\mathbb{R}^d} \left|\int_0^t \Phi_\varepsilon(X_s^\varepsilon(x)) \, \mathrm{d}s\right| \mathrm{d}\gamma_d(x) \\
&\quad =: I_1 + I_2. \qquad (3.7)
\end{aligned}$$

Using Burkholder's inequality, we get

$$\mathbb{E}\left|\sum_{j=1}^m \int_0^t \delta(A_j^\varepsilon)(X_s^\varepsilon(x)) \, \mathrm{d}w_s^j\right| \leq 2\,\mathbb{E}\left[\left(\sum_{j=1}^m \int_0^t \big|\delta(A_j^\varepsilon)(X_s^\varepsilon(x))\big|^2 \, \mathrm{d}s\right)^{1/2}\right].$$

For the sake of simplifying the notations, write $\Psi_\varepsilon = \sum_{j=1}^m |\delta(A_j^\varepsilon)|^2$. By Cauchy's inequality,

$$I_1 \le 2\bigg[\int_0^t \mathbb{E}\int_{\mathbb{R}^d} \big|\Psi_\varepsilon(X_s^\varepsilon(x))\big|\,\mathrm{d}\gamma_d(x)\mathrm{d}s\bigg]^{1/2}. \tag{3.8}$$

Now we are going to estimate $\mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon(X_s^\varepsilon(x))\big|^{2^\alpha}\mathrm{d}\gamma_d(x)$ for $\alpha\in\mathbb{Z}_+$ which will be done inductively. First if $s\in[0,T_0]$, then by (3.4) and (3.6), along with Cauchy's inequality,

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon(X_s^\varepsilon(x))\big|^{2^\alpha}\,\mathrm{d}\gamma_d(x) &= \mathbb{E}\int_{\mathbb{R}^d}|\Psi_\varepsilon(y)|^{2^\alpha}K_s^\varepsilon(y)\,\mathrm{d}\gamma_d(y)\\ &\le \|\Psi_\varepsilon\|_{L^{2^{\alpha+1}}(\gamma_d)}^{2^\alpha}\|K_s^\varepsilon\|_{L^2(\mathbb{P}\times\gamma_d)}\\ &\le \Lambda_{T_0}\|\Psi_\varepsilon\|_{L^{2^{\alpha+1}}(\gamma_d)}^{2^\alpha}.\end{aligned} \tag{3.9}$$

Now for $s\in\,]T_0,2T_0]$, we shall use the flow property of X_s^ε: let $(\theta_{T_0}w)_t := w_{T_0+t}-w_{T_0}$ and X_t^{ε,T_0} be the solution of the Itô SDE driven by the new Brownian motion $(\theta_{T_0}w)_t$, then

$$X_{T_0+t}^\varepsilon(x,w) = X_t^{\varepsilon,T_0}\big(X_{T_0}^\varepsilon(x,w),\theta_{T_0}w\big),\quad \text{for all } t\ge 0,$$

and X_t^{ε,T_0} enjoys the same properties as X_t^ε. Therefore,

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon(X_s^\varepsilon(x))\big|^{2^\alpha}\,\mathrm{d}\gamma_d(x) &= \mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon\big(X_{s-T_0}^{\varepsilon,T_0}(X_{T_0}^\varepsilon(x))\big)\big|^{2^\alpha}\,\mathrm{d}\gamma_d(x)\\ &= \mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon\big(X_{s-T_0}^{\varepsilon,T_0}(y)\big)\big|^{2^\alpha}K_{T_0}^\varepsilon(y)\,\mathrm{d}\gamma_d(y)\end{aligned}$$

which is dominated, using Cauchy-Schwarz inequality

$$\begin{aligned}&\bigg(\mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon\big(X_{s-T_0}^{\varepsilon,T_0}(y)\big)\big|^{2^{\alpha+1}}\,\mathrm{d}\gamma_d(y)\bigg)^{1/2}\|K_{T_0}^\varepsilon\|_{L^2(\mathbb{P}\times\gamma_d)}\\ &\quad\le \Big(\Lambda_{T_0}\|\Psi_\varepsilon\|_{L^{2^{\alpha+2}}(\gamma_d)}^{2^{\alpha+1}}\Big)^{1/2}\Lambda_{T_0} = \Lambda_{T_0}^{1+2^{-1}}\|\Psi_\varepsilon\|_{L^{2^{\alpha+2}}(\gamma_d)}^{2^\alpha}.\end{aligned}$$

Repeating this procedure, we finally obtain, for all $s\in[0,T]$,

$$\mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon(X_s^\varepsilon(x))\big|^{2^\alpha}\,\mathrm{d}\gamma_d(x)\le \Lambda_{T_0}^{1+2^{-1}+\ldots+2^{-N+1}}\|\Psi_\varepsilon\|_{L^{2^{\alpha+N}}(\gamma_d)}^{2^\alpha},$$

where $N\in\mathbb{Z}_+$ is the unique integer such that $(N-1)T_0 < T\le NT_0$. In particular, taking $\alpha=0$ gives

$$\mathbb{E}\int_{\mathbb{R}^d}\big|\Psi_\varepsilon(X_s^\varepsilon(x))\big|\mathrm{d}\gamma_d(x)\le \Lambda_{T_0}^2\|\Psi_\varepsilon\|_{L^{2^N}(\gamma_d)}. \tag{3.10}$$

By Lemma 3.2,

$$\sup_{0<\varepsilon\le 1}\|\Psi_\varepsilon\|_{L^{2^N}(\gamma_d)}\le \bigg\|2\sum_{j=1}^m\big(|A_j|^2+e^2|\delta(A_j)|^2\big)\bigg\|_{L^{2^N}(\gamma_d)} =: C_1$$

whose right hand side is finite under the assumptions (A2)–(A4). This along with (3.8) and (3.10) leads to

$$I_1\le 2\,(C_1T)^{1/2}\Lambda_{T_0}. \tag{3.11}$$

The same manipulation works for the term I_2 and we get

$$I_2\le C_2T\Lambda_{T_0}^2, \tag{3.12}$$

where

$$C_2 = \left\| |A_0| + e|\delta(A_0)| + \frac{3}{2}\sum_{j=1}^m |A_j|^2 + \sum_{j=1}^m |\nabla A_j|^2 \right\|_{L^{2^N}(\gamma_d)} < \infty.$$

Now we draw the conclusion from (3.7), (3.11) and (3.12). $\square$

It follows from Theorem 3.3 that the family $\{K^\varepsilon_\cdot\}_{0<\varepsilon\le 1}$ is weakly compact in $L^1([0,T]\times\Omega\times\mathbb{R}^d)$. Along a subsequence, $K^\varepsilon_\cdot$ converges weakly to some $K_\cdot \in L^1([0,T]\times\Omega\times\mathbb{R}^d)$ as $\varepsilon\to 0$. Let

$$\mathcal{C} = \left\{ u \in L^1([0,T]\times\Omega\times\mathbb{R}^d):\ u_t \ge 0,\ \int_{\mathbb{R}^d} \mathbb{E}(u_t \log u_t)\,\mathrm{d}\gamma_d \le 2\,(C_1 T)^{1/2}\Lambda_{T_0} + C_2 T \Lambda_{T_0}^2 \right\}.$$

By convexity of the function $s \to s\log s$, it is clear that $\mathcal{C}$ is a convex subset of $L^1([0,T]\times\Omega\times\mathbb{R}^d)$. Since the weak closure of $\mathcal{C}$ coincides with the strong one, there exists a sequence of functions $u^{(n)} \in \mathcal{C}$ which converges to K in $L^1([0,T]\times\Omega\times\mathbb{R}^d)$. Along a subsequence, $u^{(n)}$ converges to K almost everywhere. Hence by Fatou's lemma, we get for almost all $t\in[0,T]$,

$$\int_{\mathbb{R}^d} \mathbb{E}(K_t \log K_t)\,\mathrm{d}\gamma_d \le 2\,(C_1 T)^{1/2}\Lambda_{T_0} + C_2 T \Lambda_{T_0}^2. \tag{3.13}$$

Theorem 3.4. *Assume conditions* (A1)–(A4) *and that pathwise uniqueness holds for* SDE (1.1). *Then for each $t>0$, there is a full subset $\Omega_t \subset \Omega$ such that for all $w\in\Omega_t$, the density $\hat{K}_t$ of $(X_t)_\#\gamma_d$ with respect to γ_d exists and $\hat{K}_t \in L^1 \log L^1$.*

Proof. Under these assumptions, we can use Theorem A in [18]. For the convenience of the reader, we include the statement:

Theorem 3.5 ([18]). *Let $\sigma_n(x)$ and $b_n(x)$ be continuous, taking values respectively in the space of $(d\times m)$-matrices and $\mathbb{R}^d$. Suppose that*

$$\sup_n \big(\|\sigma_n(x)\| + |b_n(x)|\big) \le C\,(1+|x|),$$

and for any $R>0$,

$$\lim_{n\to+\infty} \sup_{|x|\le R} \big(\|\sigma_n(x)-\sigma(x)\| + |b_n(x)-b(x)|\big) = 0.$$

Suppose further that for the same Brownian motion $B(t)$, $X_n(x,t)$ solves the SDE

$$\mathrm{d}X_n(t) = \sigma_n(X_n(t))\,\mathrm{d}B(t) + b_n(X_n(t))\,\mathrm{d}t, \quad X_n(0)=x.$$

If pathwise uniqueness holds for

$$\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}B(t) + b(X(t))\,\mathrm{d}t, \quad X(0)=x,$$

then for any $R>0$, $T>0$,

$$\lim_{n\to+\infty} \sup_{|x|\le R} \mathbb{E}\left(\sup_{0\le t\le T} |X_n(t,x)-X(t,x)|^2 \right) = 0. \tag{3.14}$$

We continue the proof of Theorem 3.4. By means of Lemma 3.1 and Theorem 3.5, for any $T,R>0$, we get

$$\lim_{\varepsilon\to 0} \sup_{|x|\le R} \mathbb{E}\left(\sup_{0\le t\le T} |X_t^\varepsilon(x)-X_t(x)|^2 \right) = 0. \tag{3.15}$$

Now fixing arbitrary $\xi \in L^\infty(\Omega)$ and $\psi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d} |\xi(\cdot)| \left|\psi(X_t^\varepsilon(x)) - \psi(X_t(x))\right| \mathrm{d}\gamma_d(x) \\
&\quad \le \|\xi\|_\infty \bigg(\int_{B(R)} + \int_{B(R)^c}\bigg) \mathbb{E}\,|\psi(X_t^\varepsilon(x)) - \psi(X_t(x))|\, \mathrm{d}\gamma_d(x) \\
&\quad =: J_1 + J_2. \qquad (3.16)
\end{aligned}$$

By (3.15),

$$\begin{aligned}
J_1 &\le \|\xi\|_\infty \|\nabla\psi\|_\infty \int_{B(R)} \mathbb{E}\,|X_t^\varepsilon(x) - X_t(x)|\, \mathrm{d}\gamma_d(x) \\
&\le \|\xi\|_\infty \|\nabla\psi\|_\infty \left[\sup_{|x|\le R} \mathbb{E}\bigg(\sup_{0\le t\le T} |X_t^\varepsilon(x) - X_t(x)|^2\bigg)\right]^{1/2} \to 0, \qquad (3.17)
\end{aligned}$$

as ε tends to 0. It is obvious that

$$J_2 \le 2\,\|\xi\|_\infty\, \|\psi\|_\infty\, \gamma_d(B(R)^c). \qquad (3.18)$$

Combining (3.16), (3.17) and (3.18), we obtain

$$\limsup_{\varepsilon\to 0} \mathbb{E}\int_{\mathbb{R}^d} |\xi| \left|\psi(X_t^\varepsilon(x)) - \psi(X_t(x))\right| \mathrm{d}\gamma_d(x) \le 2\,\|\xi\|_\infty\, \|\psi\|_\infty\, \gamma_d(B(R)^c) \to 0$$

as $R \uparrow \infty$. Therefore

$$\lim_{\varepsilon\to 0} \mathbb{E}\int_{\mathbb{R}^d} \xi\, \psi(X_t^\varepsilon(x))\, \mathrm{d}\gamma_d(x) = \mathbb{E}\int_{\mathbb{R}^d} \xi\, \psi(X_t(x))\, \mathrm{d}\gamma_d. \qquad (3.19)$$

On the other hand, by Theorem 3.3, for each fixed $t \in [0,T]$, up to a subsequence, K_t^ε converges weakly in $L^1(\Omega \times \mathbb{R}^d)$ to some $\hat{K}_t$, hence

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^d} \xi\, \psi(X_t^\varepsilon(x))\, \mathrm{d}\gamma_d(x) &= \mathbb{E}\int_{\mathbb{R}^d} \xi\, \psi(y) K_t^\varepsilon(y)\, \mathrm{d}\gamma_d(y) \\
&\to \mathbb{E}\int_{\mathbb{R}^d} \xi\, \psi(y) \hat{K}_t(y)\, \mathrm{d}\gamma_d(y). \qquad (3.20)
\end{aligned}$$

This together with (3.19) leads to

$$\mathbb{E}\int_{\mathbb{R}^d} \xi\, \psi(X_t(x))\, \mathrm{d}\gamma_d(x) = \mathbb{E}\int_{\mathbb{R}^d} \xi\, \psi(y) \hat{K}_t(y)\, \mathrm{d}\gamma_d(y).$$

By the arbitrariness of $\xi \in L^\infty(\Omega)$, there exists a full measure subset Ω_ψ of Ω such that

$$\int_{\mathbb{R}^d} \psi(X_t(x))\, \mathrm{d}\gamma_d(x) = \int_{\mathbb{R}^d} \psi(y) \hat{K}_t(y)\, \mathrm{d}\gamma_d(y), \quad \text{for any } \omega \in \Omega_\psi.$$

Now by the separability of $C_c^\infty(\mathbb{R}^d)$, there exists a full subset Ω_t such that the above equality holds for any $\psi \in C_c^\infty(\mathbb{R}^d)$. Hence $(X_t)_\# \gamma_d = \hat{K}_t \gamma_d$. □

Remark 3.6. *The $K_t(w,x)$ appearing in* (3.13) *is defined almost everywhere. It is easy to see that $K_t(w,x)$ is a measurable modification of $\{\hat{K}_t(w,x);\ t \in [0,T]\}$.*

Remark 3.7. *Beyond the Lipschitz condition, several sufficient conditions guaranteeing pathwise uniqueness for* SDE (1.1) *can be found in the literature. For example in* [13], *the authors give the condition*

$$\sum_{j=1}^{m} |A_j(x)-A_j(y)|^2 \le C\,|x-y|^2 r\big(|x-y|^2\big), \quad |A_0(x)-A_0(y)| \le C\,|x-y| r\big(|x-y|^2\big),$$

for $|x-y|\le c_0$ *small enough, where* $r\colon\ ]0,c_0]\to\ ]0,+\infty[$ *is* C^1 *satisfying*

(i) $\lim_{s\to 0} r(s)=+\infty$,

(ii) $\lim_{s\to 0}\dfrac{sr'(s)}{r(s)}=0$, *and*

(iii) $\displaystyle\int_0^{c_0}\frac{\mathrm{d}s}{sr(s)}=+\infty.$

Corollary 3.8. *Suppose that the vector fields* $A_0, A_1, \dots, A_m$ *are globally Lipschitz continuous and there exists a constant* $C>0$, *such that*

$$\sum_{j=1}^{m}\langle x, A_j(x)\rangle^2 \le C\,(1+|x|^2) \quad \textit{for all } x\in\mathbb{R}^d. \tag{3.21}$$

Then $(X_t)_\#\mathrm{Leb}_d \ll \mathrm{Leb}_d$ *for any* $t\in[0,T]$.

Proof. It is obvious that hypotheses (A1), (A2) and (A4) are satisfied, and that for some constant $C>0$,

$$|\delta(A_0)|(x)\le C(1+|x|^2).$$

Hence there exists $\lambda_0>0$ such that $\int_{\mathbb{R}^d}\exp\left(\lambda_0|\delta(A_0)|\right)\,\mathrm{d}\gamma_d<+\infty$. Finally we have

$$\sum_{j=1}^{m}|\,\delta(A_j)|^2(x)\le 2\sum_{j=1}^{m}\langle x, A_j(x)\rangle^2+2\sum_{j=1}^{m}\mathrm{Lip}(A_j)^2.$$

Therefore, under condition (3.21), there exists $\lambda_0>0$ such that

$$\int_{\mathbb{R}^d}\exp\left(\lambda_0\sum_{j=1}^{m}|\delta(A_j)|^2\right)\mathrm{d}\gamma_d<+\infty.$$

Hence, hypothesis (A3) is satisfied as well. By Theorem 3.4, we have $(X_t)_\#\gamma_d=\hat{K}_t\gamma_d$. Let A be a Borel subset of $\mathbb{R}^d$ such that $\mathrm{Leb}_d(A)=0$, then $\gamma_d(A)=0$; therefore $\int_{\mathbb{R}^d}\mathbf{1}_{\{X_t(x)\in A\}}\,\mathrm{d}\gamma_d(x)=0$. It follows that $\mathbf{1}_{\{X_t(x)\in A\}}=0$ for Leb_d almost every x, which implies $\mathrm{Leb}_d(X_t\in A)=0$; this means that $(X_t)_\#\mathrm{Leb}_d$ is absolutely continuous with respect to Leb_d. □

In the next section, we shall prove that under the conditions of Corollary 3.8, the density of $(X_t)_\#\mathrm{Leb}_d$ with respect to Leb_d is strictly positive, in other words, Leb_d is quasi-invariant under X_t.

Corollary 3.9. *Assume that conditions* (A1)–(A4) *hold. Let* $\sigma=(A^i_j)$ *and suppose that for some* $C>0$,

$$\sigma(x)\sigma(x)^*\ge C\,\mathrm{Id}, \quad \textit{for all } x\in\mathbb{R}^d.$$

Then $(X_t)_\#\gamma_d$ *is absolutely continuous with respect to* γ_d.

Proof. The conditions (A1)–(A4) are stronger than those in Theorem 1.1 of [34] given by X. Zhang, so the pathwise uniqueness holds. Hence Theorem 3.4 applies to this case. □

4 Quasi-invariance under stochastic flow

In the sequel, by quasi-invariance we mean that the Radon-Nikodym derivative of the corresponding push-forward measure is strictly positive. First we prove that in the situation of Corollary 3.8, the Lebesgue measure is in fact quasi-invariant under the stochastic flow of homeomorphisms. To this end, we need some preparations. In what follows, $T_0 > 0$ is chosen small enough such that (3.5) holds.

Proposition 4.1. *Let $q \geq 2$. Then*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathbb{E}\bigg(\bigg| \sup_{0\leq t\leq T_0} \sum_{j=1}^m \int_0^t \big[\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)\big] \mathrm{d}w_s^j \bigg|^q\bigg) \mathrm{d}\gamma_d = 0. \tag{4.1}$$

Proof. By Burkholder's inequality,

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\leq t\leq T_0} \bigg| \sum_{j=1}^m \int_0^t \big[\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)\big]\, \mathrm{d}w_s^j \bigg|^q\bigg) \\
&\quad\leq C\, \mathbb{E}\bigg[\bigg(\int_0^{T_0} \sum_{j=1}^m |\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)|^2\, \mathrm{d}s\bigg)^{q/2}\bigg] \\
&\quad\leq C\, T_0^{q/2-1} \sum_{j=1}^m \int_0^{T_0} \mathbb{E}\big(|\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q\big)\, \mathrm{d}s.
\end{aligned}$$

Again by the inequality $(a+b)^q \leq C_q\,(a^q + b^q)$, there exists a constant $C_{q,T_0} > 0$ such that the above quantity is dominated by

$$C_{q,T_0} \sum_{j=1}^m \bigg[\int_0^{T_0} \mathbb{E}\big(|\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s^\varepsilon)|^q\big)\mathrm{d}s + \int_0^{T_0} \mathbb{E}\big(|\delta(A_j)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q\big)\mathrm{d}s\bigg]. \tag{4.2}$$

Let I_1^ε and I_2^ε be the two terms in the squared bracket of (4.2). Note that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}\big(|\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s^\varepsilon)|^q\big)\mathrm{d}\gamma_d \\
&\quad= \mathbb{E} \int_{\mathbb{R}^d} |\delta(A_j^\varepsilon) - \delta(A_j)|^q K_s^\varepsilon\, \mathrm{d}\gamma_d \\
&\quad\leq \|\delta(A_j^\varepsilon) - \delta(A_j)\|_{L^{2q}(\gamma_d)}^q \|K_s^\varepsilon\|_{L^2(\mathbb{P}\times\gamma_d)}.
\end{aligned} \tag{4.3}$$

According to (3.5), for $s \leq T_0$, we have $\|K_s^\varepsilon\|_{L^2(\mathbb{P}\times\gamma_d)} \leq \Lambda_{T_0}$. Remark that

$$\delta(A_j^\varepsilon) = \delta(\varphi_\varepsilon P_\varepsilon A_j) = \varphi_\varepsilon\, e^\varepsilon P_\varepsilon \delta(A_j) - \langle \nabla\varphi_\varepsilon, P_\varepsilon A_j\rangle,$$

which converges to $\delta(A_j)$ in $L^{2q}(\gamma_d)$. By (4.3),

$$\begin{aligned}
\int_{\mathbb{R}^d} I_1^\varepsilon\, \mathrm{d}\gamma_d &= \int_0^{T_0} \bigg[\int_{\mathbb{R}^d} \mathbb{E}\big(|\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s^\varepsilon)|^q\big)\mathrm{d}\gamma_d\bigg]\mathrm{d}s \\
&\leq T_0 \Lambda_{T_0}\, \|\delta(A_j^\varepsilon) - \delta(A_j)\|_{L^{2q}(\gamma_d)}^q
\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$.

For the estimate of I_2^ε, we remark that $\int_{\mathbb{R}^d} |\delta(A_j)|^{2q}\, \mathrm{d}\gamma_d < +\infty$. Let $\eta > 0$ be given. There exists $\psi \in C_c(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |\delta(A_j) - \psi|^{2q}\, \mathrm{d}\gamma_d \leq \eta^2.$$

We have, for some constant $C_q > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}\big(|\delta(A_j)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q\big)\,\mathrm{d}\gamma_d \\
&\quad \le C_q\Big[\int_{\mathbb{R}^d} \mathbb{E}\big(|\delta(A_j)(X_s^\varepsilon) - \psi(X_s^\varepsilon)|^q\big)\mathrm{d}\gamma_d + \int_{\mathbb{R}^d} \mathbb{E}\big(|\psi(X_s^\varepsilon) - \psi(X_s)|^q\big)\mathrm{d}\gamma_d \\
&\qquad\quad + \int_{\mathbb{R}^d} \mathbb{E}\big(|\psi(X_s) - \delta(A_j)(X_s)|^q\big)\mathrm{d}\gamma_d\Big].
\end{aligned} \tag{4.4}$$

Again by (3.6), we find

$$\begin{aligned}
\mathbb{E}\left[\int_{\mathbb{R}^d} |\delta(A_j)(X_s^\varepsilon) - \psi(X_s^\varepsilon)|^q\,\mathrm{d}\gamma_d\right] &= \mathbb{E}\left[\int_{\mathbb{R}^d} |\delta(A_j) - \psi|^q\, K_s^\varepsilon\,\mathrm{d}\gamma_d\right] \\
&\le \|\delta(A_j) - \psi\|^q_{L^{2q}(\gamma_d)}\,\Lambda_{T_0} \le \Lambda_{T_0}\eta.
\end{aligned}$$

In the same way,

$$\mathbb{E}\Big[\int_{\mathbb{R}^d} |\delta(A_j)(X_s) - \psi(X_s)|^q\,\mathrm{d}\gamma_d\Big] \le \Lambda_{T_0}\eta.$$

To estimate the second term on the right hand side of (4.4), we use Theorem 3.5: from (3.14), we see that up to a subsequence, $X_s^\varepsilon(w,x)$ converges to $X_s(w,x)$, for each $s \le T_0$ and almost all $(w,x) \in \Omega \times \mathbb{R}^d$. By Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathbb{E}\big(|\psi(X_s^\varepsilon) - \psi(X_s)|^q\big)\,\mathrm{d}\gamma_d = 0.$$

In conclusion, $\lim\limits_{\varepsilon\to 0} \int_{\mathbb{R}^d} I_2^\varepsilon\,\mathrm{d}\gamma_d = 0$. According to (4.2), the proof of (4.1) is complete. □

Proposition 4.2. *Let Φ be defined by*

$$\Phi = \delta(A_0) + \frac{1}{2}\sum_{j=1}^m |A_j|^2 + \frac{1}{2}\sum_{j=1}^m \langle \nabla A_j, (\nabla A_j)^*\rangle, \tag{4.5}$$

and analogously Φ_ε where A_j is replaced by A_j^ε. Then

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d}\int_0^{T_0} \mathbb{E}\big(|\Phi_\varepsilon(X_s^\varepsilon) - \Phi(X_s)|^q\big)\,\mathrm{d}s\mathrm{d}\gamma_d = 0. \tag{4.6}$$

Proof. Along the lines of the proof of Proposition 4.1, it is sufficient to remark that

$$\lim_{\varepsilon\to 0} \|\Phi_\varepsilon - \Phi\|_{L^{2q}(\gamma_d)} = 0. \tag{4.7}$$

To see this, let us check convergence for the last term in the definition of Φ_ε. We have

$$\begin{aligned}
&|\langle \nabla A_j^\varepsilon, (\nabla A_j^\varepsilon)^*\rangle - \langle \nabla A_j, (\nabla A_j)^*\rangle| \\
&\quad \le \|\nabla A_j^\varepsilon - \nabla A_j\|\,\|\nabla A_j^\varepsilon\| + \|\nabla A_j\|\,\|\nabla A_j^\varepsilon - \nabla A_j\|.
\end{aligned}$$

Note that $A_j^\varepsilon = \varphi_\varepsilon P_\varepsilon A_j$. Thus

$$\nabla A_j^\varepsilon = \nabla\varphi_\varepsilon \otimes P_\varepsilon A_j + e^{-\varepsilon}\varphi_\varepsilon P_\varepsilon(\nabla A_j),$$

which converges to ∇A_j in $L^{2q}(\gamma_d)$ as $\varepsilon \to 0$. □

Now we can prove

Proposition 4.3. *Under the conditions of Corollary 3.8, the Lebesgue measure* Leb_d *is quasi-invariant under the stochastic flow.*

Proof. Let k_t be the density of $(X_t)_\#\mathrm{Leb}_d$ with respect to Leb_d. We shall prove that k_t is strictly positive. Set

$$\tilde K_t^\varepsilon(x) = \exp\Big(-\sum_{j=1}^m \int_0^t \delta(A_j^\varepsilon)(X_s^\varepsilon(x))\,\mathrm{d}w_s^j - \int_0^t \Phi_\varepsilon(X_s^\varepsilon(x))\,\mathrm{d}s\Big), \tag{4.8}$$

where Φ_ε is defined in Proposition 4.2. By (2.3) we have

$$\int_{\mathbb{R}^d} \psi(X_t^\varepsilon)\tilde K_t^\varepsilon\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d} \psi\,\mathrm{d}\gamma_d, \quad \psi\in C_c^1(\mathbb{R}^d). \tag{4.9}$$

Applying Propositions 4.1 and 4.2, up to a subsequence, for each $t\le T_0$ and almost every (w,x), the term $\tilde K_t^\varepsilon(w,x)$ defined in (4.8) converges to

$$\tilde K_t(x) = \exp\Big(-\sum_{j=1}^m \int_0^t \delta(A_j)(X_s(x))\,\mathrm{d}w_s^j - \int_0^t \Phi(X_s(x))\,\mathrm{d}s\Big). \tag{4.10}$$

By Corollary 2.3 and Lemma 3.2, we may assume that T_0 is small enough so that for any $t\le T_0$, the family $\{\tilde K_t^\varepsilon : \varepsilon\le 1\}$ is also bounded in $L^2(\mathbb{P}\times\gamma_d)$. Therefore, by the uniform integrability, letting $\varepsilon\to 0$ in (4.9), we get $\mathbb{P}$-almost surely,

$$\int_{\mathbb{R}^d} \psi(X_t)\tilde K_t\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d} \psi\,\mathrm{d}\gamma_d, \quad \psi\in C_c^1(\mathbb{R}^d). \tag{4.11}$$

Now taking a Borel version of $x\to\tilde K_t(w,x)$. Under the assumptions, the solution X_t is a stochastic flow of homeomorphisms, hence the inverse flow X_t^{-1} exists. Consequently, if $t\le T_0$, we deduce from (4.11) that the density $K_t(w,x)$ of $(X_t)_\#\gamma_d$ with respect to γ_d admits the expression $K_t(w,x) = \big[\tilde K_t\big(w, X_t^{-1}(w,x)\big)\big]^{-1}$ which is strictly positive. For X_{t+T_0} with $t\le T_0$, we use the flow property: $X_{t+T_0}(w,x) = X_t(\theta_{T_0}w, X_{T_0}(w,x))$. Thus, for any $\psi\in C_c^1(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d}\psi(X_{t+T_0})\,\mathrm{d}\gamma_d &= \int_{\mathbb{R}^d}\psi\big(X_t(X_{T_0})\big)\mathrm{d}\gamma_d\\ &= \int_{\mathbb{R}^d}\psi(X_t)K_{T_0}\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d}\psi\,K_{T_0}\big(X_t^{-1}\big)K_t\,\mathrm{d}\gamma_d.\end{aligned}$$

That is to say, the density $K_{t+T_0} = K_{T_0}\big(X_t^{-1}\big)K_t$ is strictly positive. Continuing in this way, we obtain that K_t is strictly positive for any $t\ge 0$.

Now if $\rho(x)$ denotes the density of γ_d with respect to Leb_d, then

$$k_t(w,x) = \rho\big(X_t^{-1}(w,x)\big)^{-1}K_t(w,x)\rho(x) > 0$$

which concludes the proof. □

In what follows, we will give examples for which existence of the inverse flow is not known.

Theorem 4.4. *Let* $A_1,\ldots,A_m$ *be bounded* C^1 *vector fields on* $\mathbb{R}^d$ *such that their derivatives are of linear growth; furthermore let* A_0 *be continuous of linear growth such that* $\delta(A_0)$ *exists. Define*

$$\hat A_0 = A_0 - \sum_{j=1}^m \mathcal{L}_{A_j}A_j. \tag{4.12}$$

Suppose that $\delta(\hat{A}_0)$ exists and that

$$\int_{\mathbb{R}^d} \exp\big(\lambda_0\,(|\delta(A_0)| + |\delta(\hat{A}_0)|)\big)\,\mathrm{d}\gamma_d < +\infty, \quad \textit{for some } \lambda_0 > 0. \tag{4.13}$$

If pathwise uniqueness holds both for SDE (1.1) *and for*

$$\mathrm{d}Y_t = \sum_{j=1}^m A_j(Y_t)\,\mathrm{d}w_t^j - \hat{A}_0(Y_t)\,\mathrm{d}t, \tag{4.14}$$

then the solution X_t to SDE (1.1) *leaves the Gaussian measure γ_d quasi-invariant.*

Proof. Obviously the conditions in Theorem 3.4 are satisfied; hence $(X_t)_\# \gamma_d = K_t\,\gamma_d$. Let $t > 0$ be given, we consider the dual SDE to (1.1):

$$\mathrm{d}Y_s^t = \sum_{j=1}^m A_j(Y_s^t)\,\mathrm{d}w_s^{t,j} - \hat{A}_0(Y_s^t)\,\mathrm{d}s$$

for which pathwise uniqueness holds; here $w_s^t = w_{t-s} - w_t$ with $s \in [0,t]$. Let A_j^ε, $j = 0, 1, \ldots, m$, be the vector fields defined as above. Consider

$$\mathrm{d}Y_s^{t,\varepsilon} = \sum_{j=1}^m A_j^\varepsilon(Y_s^{t,\varepsilon})\,\mathrm{d}w_s^{t,j} - \hat{A}_0^\varepsilon(Y_s^{t,\varepsilon})\,\mathrm{d}s,$$

where $\hat{A}_0^\varepsilon = A_0^\varepsilon - \sum_{j=1}^m \mathcal{L}_{A_j^\varepsilon} A_j^\varepsilon$. Then it is known that $(X_t^\varepsilon)^{-1} = Y_t^{t,\varepsilon}$. It is easy to check that for some constant $C > 0$ independent of ε,

$$|\hat{A}_0^\varepsilon(x)| \le C\,(1 + |x|). \tag{4.15}$$

Moreover,

$$\mathcal{L}_{A_j^\varepsilon} A_j^\varepsilon = \sum_{k=1}^d (A_j^\varepsilon)^k \left[\frac{\partial \varphi_\varepsilon}{\partial x_k}\,P_\varepsilon A_j + \varphi_\varepsilon e^{-\varepsilon} P_\varepsilon\Big(\frac{\partial A_j}{\partial x_k}\Big)\right]$$

which converges locally uniformly to $\mathcal{L}_{A_j} A_j$. Therefore $\hat{A}_0^\varepsilon$ converges uniformly over any compact subset to $\hat{A}_0$. By Theorem 3.5,

$$\lim_{\varepsilon\to 0} \sup_{|x|\le R} \mathbb{E}\Big(\sup_{0\le s\le t} \big|Y_s^{t,\varepsilon} - Y_s^t\big|^2\Big) = 0.$$

It follows that, along a sequence, $Y_t^{t,\varepsilon}$ converges to Y_t^t for almost every (w, x). Now let $\psi_1, \psi_2 \in C_b(\mathbb{R}^d)$, we have for $t \le T_0$,

$$\int_{\mathbb{R}^d} \psi_1 \cdot \psi_2(X_t^\varepsilon)\tilde{K}_t^\varepsilon\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d} \psi_1\big(Y_t^{t,\varepsilon}\big) \cdot \psi_2\,\mathrm{d}\gamma_d.$$

Letting $\varepsilon \to 0$ leads to

$$\int_{\mathbb{R}^d} \psi_1 \cdot \psi_2(X_t)\tilde{K}_t\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d} \psi_1\big(Y_t^t\big) \cdot \psi_2\,\mathrm{d}\gamma_d. \tag{4.16}$$

Taking ψ_1 and ψ_2 positive in (4.16) and using a monotone class argument, we see that equation (4.16) holds for any positive Borel functions ψ_1 and ψ_2. Hence taking a Borel version of $\tilde{K}_t$ and setting $\psi_1 = 1/\tilde{K}_t$ in (4.16), we get

$$\int_{\mathbb{R}^d} \psi_2(X_t)\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d} \big[\tilde{K}_t(Y_t^t)\big]^{-1}\psi_2\,\mathrm{d}\gamma_d. \tag{4.17}$$

It follows that $K_t = \left[\tilde{K}_t(Y_t^t)\right]^{-1} > 0$ for $t \le T_0$. For X_{t+T_0} with $t \le T_0$, we shall use repeatedly (4.16). By the flow property, $X_{t+T_0}(w,x) = X_t(\theta_{T_0}w, X_{T_0}(w,x))$ where $(\theta_{T_0}w)_t = w_{t+T_0} - w_{T_0}$. Letting $t = T_0$ and replacing ψ_2 by $\psi_2(X_t)$ we get

$$\int_{\mathbb{R}^d} \psi_1 \cdot \psi_2(X_{t+T_0})\tilde{K}_{T_0}\,\mathrm{d}\gamma_d = \int_{\mathbb{R}^d} \psi_1\big(Y_{T_0}^{T_0}\big)\psi_2(X_t)\,\mathrm{d}\gamma_d.$$

Taking $\psi_1 = 1/\tilde{K}_{T_0}$ in the above equality, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \psi_2(X_{t+T_0})\,\mathrm{d}\gamma_d &= \int_{\mathbb{R}^d} \left[\tilde{K}_{T_0}\big(Y_{T_0}^{T_0}\big)\right]^{-1}\psi_2(X_t)\,\mathrm{d}\gamma_d \\ &= \int_{\mathbb{R}^d} \left[\tilde{K}_{T_0}\big(Y_{T_0}^{T_0}\big)\right]^{-1}\psi_2(X_t)\tilde{K}_t^{-1}\tilde{K}_t\,\mathrm{d}\gamma_d \\ &= \int_{\mathbb{R}^d} \left[\tilde{K}_{T_0}\big(Y_{T_0}^{T_0}(Y_t^t)\big)\right]^{-1}\left[\tilde{K}_t(Y_t^t)\right]^{-1}\psi_2\,\mathrm{d}\gamma_d,\end{aligned}$$

where in the last equality we have used (4.16) with $\psi_1 = \left[\tilde{K}_{T_0}\big(Y_{T_0}^{T_0}\big)\right]^{-1}\tilde{K}_t^{-1}$. It follows that the density K_{t+T_0} of $(X_{t+T_0})_{\#}\gamma_d$ with respect to γ_d is strictly positive, and so on. □

Corollary 4.5. *Let $A_1, \ldots, A_m$ be bounded C^2 vector fields such that their derivatives up to order 2 grow at most linearly, and let A_0 be a continuous vector field of linear growth. Suppose that*

$$|A_0(x) - A_0(y)| \le C_R\,|x-y|\log_k\frac{1}{|x-y|} \quad \textit{for } |x| \le R,\ |y| \le R,\ |x-y| \le c_0 \textit{ small enough,} \tag{4.18}$$

where $\log_k s = (\log s)(\log\log s)\ldots(\log\ldots\log s)$. Suppose further that

$$\mathrm{div}(A_0) = \sum_{j=1}^d \frac{\partial A_0^j}{\partial x_j}$$

exists and is bounded. Then the stochastic flow X_t defined by SDE (1.1) *leaves the Lebesgue measure quasi-invariant.*

Proof. It is obvious that $\hat{A}_0$ defined in (4.12) satisfies condition (4.18); therefore by [13], pathwise uniqueness holds for SDE (1.1) and (4.14). Note that $\delta(A_0) = \langle x, A_0\rangle - \mathrm{div}(A_0)$. Then condition (4.13) is satisfied; thus Theorem 4.4 yields the result. □

5 The case A_0 in Sobolev spaces

From now on, A_0 is not supposed to be continuous, but in some Sobolev space, that is, we replace the condition (A1) in **(H)** by

(A1') For $i = 1, \ldots, m$, $A_i \in \cap_{q\ge 1}\mathbb{D}_1^q(\gamma_d)$, $A_0 \in \mathbb{D}_1^q(\gamma_d)$ for some $q > 1$.

First we establish the following *a priori* estimate on perturbations, using the method developed in [36]. Let $\{A_0, A_1, \cdots, A_m\}$ be a family of measurable vector fields on $\mathbb{R}^d$. We shall give a precise definition of solution to the following SDE

$$\mathrm{d}X_t = \sum_{i=1}^m A_i(X_t)\,\mathrm{d}w_t^i + A_0(X_t)\,\mathrm{d}t, \quad X_0 = x. \tag{5.1}$$

Definition 5.1. *We say that a measurable map $X : \Omega \times \mathbb{R}^d \to C([0,T], \mathbb{R}^d)$ is a solution to Itô SDE* (5.1) *if*

(i) *for each $t \in [0,T]$ and almost all $x \in \mathbb{R}^d$, $w \to X_t(w,x)$ is measurable with respect to $\mathcal{F}_t$, i.e. the natural filtration generated by the Brownian motion $\{w_s;\ s \le t\}$;*

(ii) *for each $t \in [0,T]$, there exists $K_t \in L^1(\mathbb{P} \times \mathbb{R}^d)$ such that $(X_t(w,\cdot))_{\#}\gamma_d$ admits K_t as the density with respect to γ_d;*

(iii) *almost surely*

$$\sum_{i=1}^m \int_0^T |A_i(X_s(w,x))|^2 \,\mathrm{d}s + \int_0^T |A_0(X_s(w,x))| \,\mathrm{d}s < +\infty;$$

(iv) *for almost all $x \in \mathbb{R}^d$,*

$$X_t(w,x) = x + \sum_{i=1}^m \int_0^t A_i(X_s(w,x)) \,\mathrm{d}w_s^i + \int_0^t A_0(X_s(w,x)) \,\mathrm{d}s;$$

(v) *the flow property holds*

$$X_{t+s}(w,x) = X_t(\theta_s w, X_s(w,x)).$$

Now consider another family of measurable vector fields $\{\hat{A}_0, \hat{A}_1, \cdots, \hat{A}_m\}$ on $\mathbb{R}^d$, and denote by $\hat{X}_t$ the solution to the SDE

$$\mathrm{d}\hat{X}_t = \sum_{i=1}^m \hat{A}_i(\hat{X}_t) \,\mathrm{d}w_t^i + \hat{A}_0(\hat{X}_t) \,\mathrm{d}t, \quad \hat{X}_0 = x. \tag{5.2}$$

Let $\hat{K}_t$ be the density of $(\hat{X}_t)_{\#}\gamma_d$ and define

$$\Lambda_{p,T} = \sup_{0 \le t \le T} \Big(\|K_t\|_{L^p(\mathbb{P}\times\gamma_d)} \vee \|\hat{K}_t\|_{L^p(\mathbb{P}\times\gamma_d)} \Big). \tag{5.3}$$

Theorem 5.2. *Let $q > 1$. Suppose that $A_1, \cdots, A_m$ as well as $\hat{A}_1, \cdots, \hat{A}_m$ are in $\mathbb{D}_1^{2q}(\gamma_d)$ and $A_0, \hat{A}_0 \in \mathbb{D}_1^q(\gamma_d)$. Then for any $T > 0$ and $R > 0$, there exist constants $C_{d,q,R} > 0$ and $C_T > 0$ such that for any $\sigma > 0$,*

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{G_R} \log\bigg(\frac{\sup_{0\le t\le T} |X_t - \hat{X}_t|^2}{\sigma^2} + 1\bigg) \mathrm{d}\gamma_d\bigg] \\
&\quad \le C_T \Lambda_{p,T} \bigg\{ C_{d,q,R} \bigg[\|\nabla A_0\|_{L^q} + \bigg(\sum_{i=1}^m \|\nabla A_i\|_{L^{2q}}^2 \bigg)^{\frac12} + \sum_{i=1}^m \|\nabla A_i\|_{L^{2q}}^2 \bigg] \\
&\qquad + \frac{1}{\sigma^2} \sum_{i=1}^m \|A_i - \hat{A}_i\|_{L^{2q}}^2 + \frac{1}{\sigma} \bigg[\|A_0 - \hat{A}_0\|_{L^q} + \bigg(\sum_{i=1}^m \|A_i - \hat{A}_i\|_{L^{2q}}^2 \bigg)^{\frac12} \bigg] \bigg\},
\end{aligned}$$

where p is the conjugate number of q: $1/p + 1/q = 1$ and

$$G_R(w) = \Big\{ x \in \mathbb{R}^d : \sup_{0\le t\le T} |X_t(w,x)| \vee |\hat{X}_t(w,x)| \le R \Big\}. \tag{5.4}$$

Proof. Denote by $\xi_t = X_t - \hat{X}_t$, then $\xi_0 = 0$. By Itô formula,

$$\begin{aligned} \mathrm{d}|\xi_t|^2 = & 2\sum_{i=1}^m \langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle \mathrm{d}w_t^i + 2\langle \xi_t, A_0(X_t) - \hat{A}_0(\hat{X}_t)\rangle \mathrm{d}t \\ & + \sum_{i=1}^m |A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2 \mathrm{d}t. \end{aligned} \tag{5.5}$$

For $\sigma > 0$, $\log\left(\frac{|\xi_t|^2}{\sigma^2} + 1\right) = \log(|\xi_t|^2 + \sigma^2) - \log \sigma^2$. Again by the Itô formula,

$$\mathrm{d}\log(|\xi_t|^2 + \sigma^2) = \frac{\mathrm{d}|\xi_t|^2}{|\xi_t|^2 + \sigma^2} - \frac{1}{2} \cdot \frac{4\sum_{i=1}^m \langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle^2}{(|\xi_t|^2 + \sigma^2)^2} \mathrm{d}t,$$

using (5.5), we obtain

$$\begin{aligned} \mathrm{d}\log(|\xi_t|^2 + \sigma^2) = & 2\sum_{i=1}^m \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle}{|\xi_t|^2 + \sigma^2} \mathrm{d}w_t^i + 2\frac{\langle \xi_t, A_0(X_t) - \hat{A}_0(\hat{X}_t)\rangle}{|\xi_t|^2 + \sigma^2} \mathrm{d}t \\ & + \sum_{i=1}^m \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \mathrm{d}t - 2\sum_{i=1}^m \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle^2}{(|\xi_t|^2 + \sigma^2)^2} \mathrm{d}t \\ =: & \, \mathrm{d}I_1(t) + \mathrm{d}I_2(t) + \mathrm{d}I_3(t) + \mathrm{d}I_4(t). \end{aligned} \tag{5.6}$$

Let $\tau_R(x) = \inf\{t \geq 0 : |X_t(x)| \vee |\hat{X}_t(x)| > R\}$. Remark that almost surely, $G_R \subset \{x : \tau_R(x) > T\}$ and for any $t \geq 0$, $\{\tau_R > t\} \subset B(R)$. Therefore

$$\mathbb{E}\left[\int_{G_R} \sup_{0 \leq t \leq T} |I_1(t)| \, \mathrm{d}\gamma_d\right] \leq \mathbb{E}\left[\int_{B(R)} \sup_{0 \leq t \leq T \wedge \tau_R} |I_1(t)| \, \mathrm{d}\gamma_d\right].$$

By Burkholder's inequality,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T \wedge \tau_R} |I_1(t)|^2\right) \leq 4\,\mathbb{E}\left(\int_0^{T \wedge \tau_R} \sum_{i=1}^m \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle^2}{(|\xi_t|^2 + \sigma^2)^2} \, \mathrm{d}t\right),$$

which is obviously less than

$$4\,\mathbb{E}\left(\int_0^{T \wedge \tau_R} \sum_{i=1}^m \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, \mathrm{d}t\right).$$

Hence

$$\mathbb{E}\left[\int_{B(R)} \sup_{0 \leq t \leq T \wedge \tau_R} |I_1(t)| \, \mathrm{d}\gamma_d\right] \leq 4\left[\int_0^T \left(\mathbb{E}\int_{\{\tau_R > t\}} \sum_{i=1}^m \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \mathrm{d}\gamma_d\right) \mathrm{d}t\right]^{\frac{1}{2}}. \tag{5.7}$$

We have $A_i(X_t) - \hat{A}_i(\hat{X}_t) = A_i(X_t) - A_i(\hat{X}_t) + A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)$. Using the density $\hat{K}_t$, it is clear that

$$\begin{aligned} \mathbb{E}\int_{\{\tau_R > t\}} \frac{|A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \mathrm{d}\gamma_d & \leq \frac{1}{\sigma^2} \mathbb{E}\int_{\mathbb{R}^d} |A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2 \mathrm{d}\gamma_d \\ & = \frac{1}{\sigma^2} \mathbb{E}\int_{\mathbb{R}^d} |A_i - \hat{A}_i|^2 \hat{K}_t \, \mathrm{d}\gamma_d. \end{aligned}$$

Thus by Hölder's inequality and according to (5.3), we have

$$\mathbb{E}\int_{\{\tau_R > t\}} \frac{|A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \mathrm{d}\gamma_d \leq \frac{\Lambda_{p,T}}{\sigma^2} \|A_i - \hat{A}_i\|_{L^{2q}}^2. \tag{5.8}$$

Now we shall use Theorem 6.1 in the Appendix to estimate another term. Note that on the set $\{\tau_R > t\}$, $X_t, \hat{X}_t \in B(R)$, then $|X_t - \hat{X}_t| \le 2R$. Since $(X_t)_\# \gamma_d \ll \gamma_d$ and $(\hat{X}_t)_\# \gamma_d \ll \gamma_d$, we can apply (6.2) so that

$$|A_i(X_t) - A_i(\hat{X}_t)| \le C_d |X_t - \hat{X}_t| \big(M_{2R}|\nabla A_i|(X_t) + M_{2R}|\nabla A_i|(\hat{X}_t) \big).$$

Then

$$\mathbb{E}\bigg[\int_{\{\tau_R>t\}} \frac{|A_i(X_t) - A_i(\hat{X}_t)|^2}{|\xi_t|^2+\sigma^2} \mathrm{d}\gamma_d \bigg] \le C_d^2\, \mathbb{E} \int_{\{\tau_R>t\}} \big(M_{2R}|\nabla A_i|(X_t) + M_{2R}|\nabla A_i|(\hat{X}_t) \big)^2 \,\mathrm{d}\gamma_d.$$

Notice again that on $\{\tau_R(x) > t\}$, $X_t(x)$ and $\hat{X}_t(x)$ are in $B(R)$, therefore

$$\begin{aligned} \mathbb{E}\bigg[\int_{\{\tau_R>t\}} \frac{|A_i(X_t) - A_i(\hat{X}_t)|^2}{|\xi_t|^2+\sigma^2} \mathrm{d}\gamma_d \bigg] &\le 2C_d^2\, \mathbb{E} \int_{B(R)} (M_{2R}|\nabla A_i|)^2 \big(K_t + \hat{K}_t\big) \mathrm{d}\gamma_d \\ &\le 4C_d^2 \Lambda_{p,T} \bigg(\int_{B(R)} \big(M_{2R}|\nabla A_i|\big)^{2q} \mathrm{d}\gamma_d \bigg)^{\frac{1}{q}}. \end{aligned} \tag{5.9}$$

Remark that the maximal function inequality does not hold for the Gaussian measure γ_d on the whole space $\mathbb{R}^d$. However, on each ball $B(R)$,

$$\gamma_d|_{B(R)} \le \frac{1}{(2\pi)^{d/2}} \mathrm{Leb}_d|_{B(R)} \le e^{R^2/2} \gamma_d|_{B(R)}.$$

Thus, according to (6.3),

$$\begin{aligned} \int_{B(R)} \big(M_{2R}|\nabla A_i|\big)^{2q} \mathrm{d}\gamma_d &\le \frac{1}{(2\pi)^{d/2}} \int_{B(R)} \big(M_{2R}|\nabla A_i|\big)^{2q} \mathrm{d}x \le \frac{C_{d,q}}{(2\pi)^{d/2}} \int_{B(3R)} |\nabla A_i|^{2q} \mathrm{d}x \\ &\le C_{d,q} e^{9R^2/2} \int_{B(3R)} |\nabla A_i|^{2q} \mathrm{d}\gamma_d \le C_{d,q} e^{9R^2/2} \|\nabla A_i\|_{L^{2q}}^{2q}. \end{aligned}$$

Therefore by (5.9), there exists a constant $C_{d,q,R} > 0$ such that

$$\mathbb{E}\bigg[\int_{\{\tau_R>t\}} \frac{|A_i(X_t) - A_i(\hat{X}_t)|^2}{|\xi_t|^2+\sigma^2} \mathrm{d}\gamma_d \bigg] \le C_{d,q,R} \Lambda_{p,T} \|\nabla A_i\|_{L^{2q}}^2.$$

Combining this estimate with (5.7) and (5.8), we get

$$\mathbb{E}\bigg[\int_{G_R} \sup_{0\le t\le T} |I_1(t)| \mathrm{d}\gamma_d \bigg] \le C T^{\frac{1}{2}} \Lambda_{p,T}^{\frac{1}{2}} \bigg(C_{d,q,R} \sum_{i=1}^m \|\nabla A_i\|_{L^{2q}}^2 + \frac{1}{\sigma^2} \sum_{i=1}^m \|A_i - \hat{A}_i\|_{L^{2q}}^2 \bigg)^{\frac{1}{2}}. \tag{5.10}$$

Now we turn to deal with $I_2(t)$ in (5.6). We have

$$\mathbb{E}\bigg[\int_{G_R} \sup_{0\le t\le T} |I_2(t)|\, \mathrm{d}\gamma_d \bigg] \le 2 \int_0^T \bigg[\mathbb{E} \int_{G_R} \frac{|A_0(X_t) - \hat{A}_0(\hat{X}_t)|}{(|\xi_t|^2+\sigma^2)^{\frac{1}{2}}} \mathrm{d}\gamma_d \bigg] \,\mathrm{d}t.$$

Note that for $x \in G_R$, $\hat{X}_t(x) \in B(R)$ for each $t \in [0,T]$, thus

$$\mathbb{E}\bigg[\int_{G_R} \frac{|A_0(\hat{X}_t) - \hat{A}_0(\hat{X}_t)|}{(|\xi_t|^2+\sigma^2)^{\frac{1}{2}}} \mathrm{d}\gamma_d \bigg] \le \frac{1}{\sigma} \mathbb{E} \int_{B(R)} |A_0 - \hat{A}_0|\, \hat{K}_t \,\mathrm{d}\gamma_d \le \frac{\Lambda_{p,T}}{\sigma} \|A_0 - \hat{A}_0\|_{L^q}.$$

Again using (6.2),

$$\mathbb{E}\bigg[\int_{G_R}\frac{|A_0(X_t)-A_0(\hat{X}_t)|}{(|\xi_t|^2+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\gamma_d\bigg]\le C_d\,\mathbb{E}\int_{G_R}\big(M_{2R}|\nabla A_0|(X_t)+M_{2R}|\nabla A_0|(\hat{X}_t)\big)\mathrm{d}\gamma_d,$$

which is dominated by

$$C_d\,\mathbb{E}\bigg[\int_{B(R)}(M_{2R}|\nabla A_0|)\cdot(K_t+\hat{K}_t)\,d\gamma_d\bigg]\le C_{d,q,R}\,\|\nabla A_0\|_{L^q}\,\Lambda_{p,T}.$$

Therefore we get the following estimate for I_2:

$$\mathbb{E}\bigg[\int_{G_R}\sup_{0\le t\le T}|I_2(t)|\,\mathrm{d}\gamma_d\bigg]\le 2T\Lambda_{p,T}\bigg(C_{d,q,R}\|\nabla A_0\|_{L^q}+\frac{1}{\sigma}\|A_0-\hat{A}_0\|_{L^q}\bigg).\tag{5.11}$$

In the same way we have

$$\mathbb{E}\bigg[\int_{G_R}\sup_{0\le t\le T}|I_3(t)|\,\mathrm{d}\gamma_d\bigg]\le CT\Lambda_{p,T}\bigg(C_{d,q,R}\sum_{i=1}^m\|\nabla A_i\|_{L^{2q}}^2+\frac{1}{\sigma^2}\sum_{i=1}^m\|A_i-\hat{A}_i\|_{L^{2q}}^2\bigg).\tag{5.12}$$

The term $I_4(t)$ is negative and hence we omit it. Combining (5.6) and (5.10)–(5.12), we complete the proof. □

Now we shall construct a solution to SDE (5.1). To this end, we take $\varepsilon=1/n$ and we write A_j^n instead of $A_j^{1/n}$ introduced in Section 3. Then by assumption (A2) and Lemma 3.1, there is $C>0$ independent of n and i, such that

$$|A_i^n(x)|\le C(1+|x|).\tag{5.13}$$

Let X_t^n be the solution to Itô SDE (5.1) with the coefficients A_i^n ($i=0,1,\ldots,m$). Then for any $\alpha\ge 1$ and $T>0$, there exists $C_{\alpha,T}>0$ independent of n such that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_t^n|^\alpha\Big)\le C_{\alpha,T}(1+|x|^\alpha),\quad\text{for all } x\in\mathbb{R}^d.\tag{5.14}$$

Let K_t^n be the density of $(X_t^n)_\#\gamma_d$ with respect to γ_d. Under the hypotheses (A2)–(A4), there is $T_0>0$ such that (recall that p is the conjugate number of $q>1$):

$$\begin{aligned}\Lambda_{p,T_0}:=\bigg[\int_{\mathbb{R}^d}\exp\bigg(2pT_0\Big[&|A_0|+e|\delta(A_0)|\\&+\sum_{j=1}^m\big(2p|A_j|^2+|\nabla A_j|^2+2(p-1)e^2|\delta(A_j)|^2\big)\Big]\bigg)\mathrm{d}\gamma_d\bigg]^{\frac{p-1}{p(2p-1)}}<\infty.\end{aligned}\tag{5.15}$$

Similar to (3.6), we have

$$\sup_{t\in[0,T_0]}\sup_{n\ge 1}\|K_t^n\|_{L^p(\gamma_d\times\mathbb{P})}\le\Lambda_{p,T_0}<+\infty.\tag{5.16}$$

Now we shall prove that the family $\{X_\cdot^n:n\ge 1\}$ is convergent to some stochastic field.

Theorem 5.3. *Let T_0 be given in* (5.15). *Then under the assumptions* (A1') *and* (A2)–(A4), *there exists $X:\Omega\times\mathbb{R}^d\to C([0,T_0],\mathbb{R}^d)$ such that for any $\alpha\ge 1$,*

$$\lim_{n\to\infty}\mathbb{E}\bigg[\int_{\mathbb{R}^d}\Big(\sup_{0\le t\le T_0}|X_t^n-X_t|^\alpha\Big)\mathrm{d}\gamma_d\bigg]=0.\tag{5.17}$$

Proof. We shall prove that $\{X^n;\ n \geq 1\}$ is a Cauchy sequence in $L^\alpha(\Omega \times \mathbb{R}^d; C([0,T_0],\mathbb{R}^d))$. Denote by $\|\cdot\|_{\infty,T_0}$ the uniform norm on $C([0,T_0],\mathbb{R}^d)$, so what we have to prove is

$$\lim_{n,k\to+\infty} \mathbb{E}\Big(\int_{\mathbb{R}^d} \|X^n - X^k\|_{\infty,T_0}^\alpha \,\mathrm{d}\gamma_d \Big) = 0. \tag{5.18}$$

First by (5.14), the quantity

$$J_{\alpha,T_0} := \sup_{n\geq 1} \mathbb{E}\Big(\int_{\mathbb{R}^d} \|X^n\|_{\infty,T_0}^{2\alpha} \,\mathrm{d}\gamma_d \Big) \leq C_{\alpha,T_0} \int_{\mathbb{R}^d} (1+|x|^{2\alpha}) \,\mathrm{d}\gamma_d \tag{5.19}$$

is obviously finite. Let $R > 0$ and set

$$G_{n,R}(w) = \{x \in \mathbb{R}^d;\ \|X^n(w,x)\|_{\infty,T_0} \leq R\}.$$

Using (5.19), for any $\alpha \geq 1$ and $R > 0$, we have

$$\sup_{n\geq 1} \mathbb{E}\big(\gamma_d(G_{n,R}^c)\big) \leq \frac{J_{\alpha,T_0}}{R^{2\alpha}}.$$

Now by Cauchy-Schwarz inequality

$$\begin{aligned}
&\mathbb{E}\Big(\int_{G_{n,R}^c \cup G_{k,R}^c} \|X^n - X^k\|_{\infty,T_0}^\alpha \,\mathrm{d}\gamma_d \Big) \\
&\quad\leq \Big(\mathbb{E}\big[\gamma_d\big(G_{n,R}^c \cup G_{k,R}^c\big)\big] \Big)^{1/2} \cdot \Big(\mathbb{E} \int_{\mathbb{R}^d} \|X^n - X^k\|_{\infty,T_0}^{2\alpha} \,\mathrm{d}\gamma_d \Big)^{1/2} \\
&\quad\leq \Big(\frac{2J_{\alpha,T_0}}{R^{2\alpha}} \Big)^{1/2} \cdot \big(2^{2\alpha} J_{\alpha,T_0}\big)^{1/2}.
\end{aligned}$$

Let $\varepsilon > 0$ be given; choose $R > 1$ big enough such that the last quantity in the above inequality is less than ε. Then we have for any $n,k \geq 1$,

$$\mathbb{E}\Big(\int_{G_{n,R}^c \cup G_{k,R}^c} \|X^n - X^k\|_{\infty,T_0}^\alpha \,\mathrm{d}\gamma_d \Big) \leq \varepsilon. \tag{5.20}$$

Let

$$\sigma_{n,k} = \|A_0^n - A_0^k\|_{L^q} + \Big(\sum_{i=1}^m \|A_i^n - A_i^k\|_{L^{2q}}^2 \Big)^{1/2},$$

which tends to 0 as $n,k \to +\infty$ since A_0^n converges to A_0 in $L^q(\gamma_d)$ and A_i^n converges to A_i in $L^{2q}(\gamma_d)$ for $i = 1, \cdots, m$. Now applying Theorem 5.2 with A_i and $\hat{A}_i$ being replaced respectively by A_i^n and A_i^k, we get

$$\begin{aligned}
I_{n,k} &:= \mathbb{E}\Big[\int_{G_{n,R}\cap G_{k,R}} \log\Big(\frac{\|X^n - X^k\|_{\infty,T_0}^2}{\sigma_{n,k}^2} + 1 \Big) \,\mathrm{d}\gamma_d \Big] \\
&\leq C_{T_0} \Lambda_{p,T_0} \Big\{ C_{d,q,R} \Big[\|\nabla A_0^n\|_{L^q} + \Big(\sum_{i=1}^m \|\nabla A_i^n\|_{L^{2q}}^2 \Big)^{1/2} + \sum_{i=1}^n \|\nabla A_i^n\|_{L^{2q}}^2 \Big] + 2 \Big\}.
\end{aligned}$$

Recall that $A_i^n = \varphi_{1/n}\, P_{1/n} A_i$, then $\nabla A_i^n = \nabla\varphi_{1/n} \otimes P_{1/n} A_i + \varphi_{1/n}\, e^{-1/n} P_{1/n} \nabla A_i$, therefore

$$|\nabla A_i^n| \leq P_{1/n}(|A_i| + |\nabla A_i|).$$

We get the following uniform estimates

$$\|\nabla A_0^n\|_{L^q} \le \|A_0\|_{\mathbb{D}_1^q}, \quad \|\nabla A_i^n\|_{L^{2q}} \le \|A_i\|_{\mathbb{D}_1^{2q}}.$$

So the quantity $I_{n,k}$ is uniformly bounded with respect to n, k. Let $\hat{\Pi}$ be the measure on $\Omega \times \mathbb{R}^d$ defined by

$$\int_{\Omega\times\mathbb{R}^d} \psi(w,x)\,\mathrm{d}\hat{\Pi}(w,x) = \mathbb{E}\Big[\int_{G_{n,R}\cap G_{k,R}} \psi(w,x)\,\mathrm{d}\gamma_d(x)\Big].$$

We have $\hat{\Pi}(\Omega \times \mathbb{R}^d) \le 1$. Let $\eta > 0$, consider

$$\Sigma_{n,k} = \{(w,x);\ \|X^n(w,x) - X^k(w,x)\|_{\infty,T_0} \ge \eta\},$$

which is equal to

$$\Big\{(w,x);\ \log\Big(\frac{\|X^n - X^k\|_{\infty,T_0}^2}{\sigma_{n,k}^2} + 1\Big) \ge \log\Big(\frac{\eta^2}{\sigma_{n,k}^2} + 1\Big)\Big\}.$$

It follows that as $n, k \to +\infty$,

$$\hat{\Pi}(\Sigma_{n,k}) \le \frac{I_{n,k}}{\log\Big(\frac{\eta^2}{\sigma_{n,k}^2} + 1\Big)} \to 0, \tag{5.21}$$

since $\sigma_{n,k} \to 0$ and the family $\{I_{n,k}; n, k \ge 1\}$ is bounded. Now

$$\begin{aligned}\mathbb{E}\Big(\int_{G_{n,R}\cap G_{k,R}} \|X^n - X^k\|_{\infty,T_0}^\alpha\,\mathrm{d}\gamma_d\Big) &= \int_{\Omega\times\mathbb{R}^d} \|X^n - X^k\|_{\infty,T_0}^\alpha\,\mathrm{d}\hat{\Pi}\\ &= \int_{\Sigma_{n,k}^c} \|X^n - X^k\|_{\infty,T_0}^\alpha\,\mathrm{d}\hat{\Pi} + \int_{\Sigma_{n,k}} \|X^n - X^k\|_{\infty,T_0}^\alpha\,\mathrm{d}\hat{\Pi}. \end{aligned}\tag{5.22}$$

The first term on the right side of (5.22) is less than η^α, while the second one, due to (5.19) and (5.21), is dominated by

$$\sqrt{\hat{\Pi}(\Sigma_{n,k})}\cdot\sqrt{\mathbb{E}\int_{\mathbb{R}^d} \|X^n - X^k\|_{\infty,T_0}^{2\alpha}\mathrm{d}\gamma_d} \le 2^\alpha\sqrt{J_{\alpha,T_0}\hat{\Pi}(\Sigma_{n,k})} \to 0 \quad \text{as } n,k \to +\infty.$$

Now taking $\eta = \varepsilon^{1/\alpha}$ and combining (5.20) and (5.22), we prove that

$$\limsup_{n,k\to+\infty} \mathbb{E}\Big[\int_{\mathbb{R}^d} \|X^n - X^k\|_{\infty,T_0}^\alpha\,\mathrm{d}\gamma_d\Big] \le 2\varepsilon,$$

which implies (5.18).

Let $X \in L^\alpha(\Omega \times \mathbb{R}^d; C([0,T_0], \mathbb{R}^d))$ be the limit of X^n in this space. We see that for each $t \in [0,T]$ and almost all $x \in \mathbb{R}^d$, $w \to X_t(w,x)$ is in $\mathcal{F}_t$. □

Proposition 5.4. *There exists a family* $\{\hat{K}_t; t \in [0,T_0]\}$ *of density functions on* $\mathbb{R}^d$ *such that* $(X_t)_\#\gamma_d = \hat{K}_t\gamma_d$ *for each* $t \in [0,T_0]$*. Moreover,* $\sup_{0\le t\le T_0} \|\hat{K}_t\|_{L^p(\mathbb{P}\times\gamma_d)} \le \Lambda_{p,T_0}$*, where* Λ_{p,T_0} *is given in* (5.16)*.*

Proof. It is the same as the proof of Theorem 3.4. □

The same arguments in the proof of Proposition 4.1 and 4.2 yield the following

Proposition 5.5. *For any $\alpha \geq 2$, up to a subsequence,*

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\mathbb{E}\Big(\sup_{0\leq t\leq T_0}\Big|\sum_{i=1}^m\int_0^t\big[A_i^n(X_s^n)-A_i(X_s)\big]\mathrm{d}w_s^i\Big|^\alpha\Big)\mathrm{d}\gamma_d=0,$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\Big[\mathbb{E}\int_0^{T_0}|A_0^n(X_s^n)-A_0(X_s)|^\alpha\mathrm{d}s\Big]\mathrm{d}\gamma_d=0.$$

Now for regularized vector fields $A_i^n, i=0,1,\cdots,m$, we have

$$X_t^n(x)=x+\sum_{i=1}^m\int_0^t A_i^n(X_s^n)\,\mathrm{d}w_s^i+\int_0^t A_0^n(X_s^n)\,\mathrm{d}s. \tag{5.23}$$

When $n\to+\infty$, by Theorem 5.3 and Proposition 5.5, the two sides of (5.23) converge respectively to X and

$$x+\sum_{i=1}^m\int_0^{\cdot} A_i(X_s)\,\mathrm{d}w_s^i+\int_0^{\cdot} A_0(X_s)\,\mathrm{d}s$$

in the space $L^\alpha(\Omega\times\mathbb{R}^d;C([0,T_0],\mathbb{R}^d))$. Therefore for almost all $x\in\mathbb{R}^d$, the following equality holds $\mathbb{P}$-almost surely:

$$X_t(x)=x+\sum_{i=1}^m\int_0^t A_i(X_s)\,\mathrm{d}w_s^i+\int_0^t A_0(X_s)\,\mathrm{d}s,\quad \text{for all } t\in[0,T_0].$$

That is to say, X_t solves SDE (5.1) over $[0,T_0]$.

The following result proves the pathwise uniqueness to SDE (5.1) for a.e. initial value $x\in\mathbb{R}^d$.

Proposition 5.6. *Under the conditions* (A1') *and* (A2)–(A4)*, the SDE* (5.1) *has a unique solution on the interval* $[0,T_0]$.

Proof. Let $(Y_t)_{t\in[0,T_0]}$ be another solution. Set, for $R>0$,

$$G_R=\Big\{(w,x)\in\Omega\times\mathbb{R}^d;\ \sup_{0\leq t\leq T_0}|X_t(w,x)-Y_t(w,x)|\leq R\Big\}.$$

Remark that in Theorem 5.2, the terms involving $1/\sigma$ and $1/\sigma^2$ are equal to zero. Therefore the term

$$\begin{aligned}I&:=\mathbb{E}\int_{G_R}\log\Big(\frac{\sup_{0\leq t\leq T_0}|X_t-Y_t|^2}{\sigma^2}+1\Big)\mathrm{d}\gamma_d\\&\leq C_{T_0}\Lambda_{p,T_0}C_{d,q,R}\Big[\|A_0\|_{\mathbb{D}_1^q}+\Big(\sum_{i=1}^m\|A_i\|_{\mathbb{D}_1^{2q}}^2\Big)^{\frac{1}{2}}+\sum_{i=1}^m\|A_i\|_{\mathbb{D}_1^{2q}}^2\Big]\end{aligned}$$

is bounded for any $\sigma>0$. Consider for $\eta>0$,

$$\Sigma_\eta=\Big\{(w,x);\ \sup_{0\leq t\leq T_0}|X_t(w,x)-Y_t(w,x)|\geq\eta\Big\}.$$

Similar to (5.21), we have

$$\mathbb{E}\Big(\int_{G_R}\mathbf{1}_{\Sigma_\eta}\mathrm{d}\gamma_d\Big)\leq\frac{I}{\log\big(\frac{\eta^2}{\sigma^2}+1\big)}\to 0$$

as $\sigma \to 0$. So we obtain

$$\mathbf{1}_{G_R} \cdot \sup_{0\le t\le T_0} |X_t - Y_t| = 0, \quad (\mathbb{P}\times\gamma_d)\text{-a.s.}$$

Letting $R \to \infty$, we obtain that $(\mathbb{P}\times\gamma_d)$ almost surely, $X_t = Y_t$ for all $t \in [0, T_0]$. $\square$

Now we extend the solution to any time interval $[0, T]$. Let $\theta_{T_0} w$ be the time-shift of the Brownian motion w and denote by $X_t^{T_0}$ the corresponding solution to SDE driven by $\theta_{T_0} w$. By Proposition 5.6, $\{X_t^{T_0}(\theta_{T_0} w, x) : 0 \le t \le T_0\}$ is the unique solution to the SDE over $[0, T_0]$:

$$X_t^{T_0}(x) = x + \sum_{i=1}^m \int_0^t A_i(X_s^{T_0}(x))\, \mathrm{d}(\theta_{T_0} w)_s^i + \int_0^t A_0(X_s^{T_0}(x))\, \mathrm{d}s.$$

For $t \in [0, T_0]$, define $X_{t+T_0}(w, x) = X_t^{T_0}(\theta_{T_0} w, X_{T_0}(w, x))$. Note that X_t is well defined on the interval $[0, 2T_0]$ up to a $(\mathbb{P}\times\gamma_d)$-negligible subset of $\Omega \times \mathbb{R}_d$. Replacing x by $X_{T_0}(x)$ in the above equation, we get easily

$$X_{t+T_0}(x) = x + \sum_{i=1}^m \int_0^{t+T_0} A_i(X_s(x))\, \mathrm{d}w_s^i + \int_0^{t+T_0} A_0(X_s(x))\, \mathrm{d}s.$$

Therefore X_t defined as above is a solution to SDE on the interval $[0, 2T_0]$. Continuing in this way, we obtain the solution of SDE (5.1) on $[0, T]$.

Theorem 5.7. *The $\{X_t;\ t \in [0, T]\}$ constructed above is the unique solution to SDE* (5.1) *in the sense of Definition 5.1. Moreover for each $t \in [0, T]$, the density K_t of $(X_t)_{\#}\gamma_d$ with respect to γ_d is in the space $L^1 \log L^1$.*

Proof. Let Y_t, $t \in [0, T]$ be another solution in the sense of Definition 5.1. First by Proposition 5.6, we have $(\mathbb{P}\times\gamma_d)$-almost surely, $Y_t = X_t$ for all $t \in [0, T_0]$. In particular, $Y_{T_0} = X_{T_0}$. Next by the flow property, Y_{t+T_0} satisfies the following equation:

$$Y_{t+T_0}(x) = Y_{T_0}(x) + \sum_{i=1}^m \int_0^t A_i\big(Y_{s+T_0}(x)\big)\, \mathrm{d}(\theta_{T_0} w)_s^i + \int_0^t A_0\big(Y_{s+T_0}(x)\big) \mathrm{d}s,$$

that is, Y_{t+T_0} is a solution with initial value Y_{T_0}. But by the above discussion, X_{t+T_0} is also a solution with the same initial value $X_{T_0} = Y_{T_0}$. Again by Proposition 5.6, we have $(\mathbb{P}\times\gamma_d)$-almost surely, $X_{t+T_0} = Y_{t+T_0}$ for all $t \le T_0$. Hence we have proved $X|_{[0,2T_0]} = Y|_{[0,2T_0]}$. Repeating this procedure, we obtain the uniqueness over $[0, T]$. The existence of density K_t of $(X_t)_{\#}\gamma_d$ with respect to γ_d beyond T_0 is deduced from the flow property. However, to insure that $K_t \in L^1 \log L^1$, we have to use Theorem 3.3 and the following

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \mathbb{E}\bigg(\sup_{0\le t\le T} |X_t^n - X_t|^\alpha \bigg) \mathrm{d}\gamma_d = 0,$$

which can be checked using the same arguments as in the proof of Propositions 4.1 and 4.2. $\square$

6 Appendix

For any locally integrable function $f \in L^1_{loc}(\mathbb{R}^d)$ and $R > 0$, the local maximal function $M_R f$ is defined by

$$M_R f(x) = \sup_{0<r\le R} \frac{1}{\mathrm{Leb}_d(B(x, r))} \int_{B(x,r)} |f(y)|\, \mathrm{d}y, \tag{6.1}$$

where $B(x, r) = \{y \in \mathbb{R}^d; |y - x| \le r\}$. The following result is the starting point for the approach concerning Sobolev coefficients, used in [5] and [36].

Theorem 6.1. *Let $f \in L^1_{loc}(\mathbb{R}^d)$ be such that $\nabla f \in L^1_{loc}(\mathbb{R}^d)$. Then there is a constant $C_d > 0$ (independent of f) and a negligible subset N, such that for $x, y \in N^c$ with $|x-y| \le R$,*

$$|f(x) - f(y)| \le C_d |x-y| \big((M_R|\nabla f|)(x) + (M_R|\nabla f|)(y) \big); \tag{6.2}$$

moreover for $p > 1$ and $f \in L^p_{loc}(\mathbb{R}^d)$, there is a constant $C_{d,p} > 0$ such that

$$\int_{B(r)} (M_R f)^p \, \mathrm{d}x \le C_{d,p} \int_{B(r+R)} |f|^p \, \mathrm{d}x. \tag{6.3}$$

Since the inequality (6.2) played a key role in the proof of Theorem 5.2, we give here its proof for the sake of the reader's convenience.

We follow the idea of the proof of Claim #2 on p.253 in [9]. For any bounded measurable subset U in $\mathbb{R}^d$ such that its Lebesgue measure $\mathrm{Leb}_d(U) > 0$, define the average of $f \in L^1_{loc}(\mathbb{R}^d)$ on U by

$$(f)_U = \fint_U f(y) \, \mathrm{d}y := \frac{1}{\mathrm{Leb}_d(U)} \int_U f(y) \, \mathrm{d}y.$$

Write $(f)_{x,r}$ instead of $(f)_{B(x,r)}$ for simplicity. Then $M_R f(x) = \sup_{0<r\le R} (|f|)_{x,r}$. We will need the following simple inequality: for any $C \in \mathbb{R}$,

$$|(f)_U - C| \le \fint_U |f(y) - C| \, \mathrm{d}y. \tag{6.4}$$

First, for any $x \in \mathbb{R}^d$ and $r \in]0, R]$, by Poincaré's inequality with $p = 1$ and $p^* = d/(d-1)$ (see [9] p.141), there is $C_d > 0$ such that

$$\begin{aligned} \fint_{B(x,r)} |f - (f)_{x,r}| \, \mathrm{d}y &\le \bigg(\fint_{B(x,r)} |f - (f)_{x,r}|^{d/(d-1)} \, \mathrm{d}y \bigg)^{(d-1)/d} \\ &\le C_d \, r \fint_{B(x,r)} |\nabla f| \, \mathrm{d}y \le C_d M_R |\nabla f|(x) \, r. \end{aligned} \tag{6.5}$$

In particular, for all $k \ge 0$, by (6.4) and (6.5),

$$\begin{aligned} |(f)_{x,r/2^{k+1}} - (f)_{x,r/2^k}| &\le \fint_{B(x,r/2^{k+1})} |f - (f)_{x,r/2^k}| \, \mathrm{d}y \\ &\le 2^d \fint_{B(x,r/2^k)} |f - (f)_{x,r/2^k}| \, \mathrm{d}y \\ &\le 2^d C_d M_R |\nabla f|(x) \, r/2^k. \end{aligned}$$

Since $f \in L^1_{loc}(\mathbb{R}^d)$, there is a negligible subset $N \subset \mathbb{R}^d$, such that for all $x \in N^c$, $f(x) = \lim_{r\to 0} (f)_{x,r}$. Thus for any $x \in N^c$, we have by summing up the above inequality that

$$|f(x) - (f)_{x,r}| \le \sum_{k=0}^{\infty} |(f)_{x,r/2^{k+1}} - (f)_{x,r/2^k}| \le 2^{1+d} C_d M_R |\nabla f|(x) \, r. \tag{6.6}$$

Next for all $x, y \in N^c, x \ne y$ and $|x-y| \le R$, let $r = |x-y|$. Then by the triangular inequality, (6.4) and (6.5),

$$\begin{aligned} |(f)_{x,r} - (f)_{y,r}| &\le \fint_{B(x,r)\cap B(y,r)} \big(|(f)_{x,r} - f(z)| + |f(z) - (f)_{y,r}| \big) \mathrm{d}z \\ &\le \tilde{C}_d \bigg[\fint_{B(x,r)} |(f)_{x,r} - f(z)| \, \mathrm{d}z + \fint_{B(y,r)} |f(z) - (f)_{y,r}| \, \mathrm{d}z \bigg] \end{aligned}$$

$$\leq \tilde{C}_d C_d \big(M_R|\nabla f|(x) + M_R|\nabla f|(y)\big) r. \tag{6.7}$$

Now (6.2) follows from the triangular inequality and (6.6), (6.7):

$$\begin{aligned}|f(x)-f(y)| &\leq |f(x)-(f)_{x,r}| + |(f)_{x,r}-(f)_{y,r}| + |(f)_{y,r}-f(y)|\\ &\leq 2^{1+d} C_d M_R|\nabla f|(x)\, r + \tilde{C}_d C_d\big(M_R|\nabla f|(x) + M_R|\nabla f|(y)\big) r\\ &\quad + 2^{1+d} C_d M_R|\nabla f|(y)\, r\\ &= C_d(2^{1+d}+\tilde{C}_d)|x-y|\big(M_R|\nabla f|(x) + M_R|\nabla f|(y)\big).\end{aligned}$$

We obtain (6.2). □

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